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SUR LES FONCTIONNELLES BILINÉAIRES*

PAR

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INTRODUCTION

Le présent travail est le développement de la dernière partie d'une note, "Sur les fonctionnelles continues," insérée dans les Comptes Rendus de l'Académie des Sciences.† Il est consacré à la représentation analytique des fonctionnelles bilinéaires. Après un rappel de quelques définitions et propriétés (§§ 1, 2) je montre d'abord (§§ 3, 4) que la continuité de la fonctionnelle distributive $U_{f,\phi}$ par rapport à f et ϕ séparément entraîne sa continuité par rapport à l'ensemble de f et de ϕ. On déduit alors facilement (§ 5) d'un théorème de F. Riesz sur les fonctionnelles linéaires la représentation d'une fonctionnelle bilinéaire par deux intégrales successives effectuées au sens de Stieltjes. Celles-ci peuvent être représentées (§ 6) par une expression de la forme

$$\lim_{\epsilon,\epsilon'=0} \sum_i \sum_j f(\xi_i)\phi(\eta_j)\Delta_{ij}u(s,t) \tag{1}$$

où Δ_{ij} est la différence seconde de u

$$\Delta_{i,j}u(s,t) = u(s_i,t_j) - u(s_{i-1},t_j) - u(s_i,t_{j-1}) + u(s_{i-1},t_{j-1})$$

et où ϵ, ϵ' sont les plus grands des intervalles $(s_i - s_{i-1})$, $(t_j - t_{j-1})$. La fonction $u(s,t)$ est à variation totale bornée par rapport à l'ensemble de s et de t (§ 7). Mais, m'écartant de la définition bien naturelle que j'avais adoptée ailleurs‡ je définis cette variation totale comme la borne supérieure de l'expression

$$\left|\sum_i \sum_j \epsilon_i \epsilon'_j \Delta_{i,j}u(s,t)\right|$$

où les ϵ_i, ϵ'_j sont de signes quelconques mais en valeurs absolues égales à l'unité. Je démontre alors que réciproquement si $u(s,t)$ est une fonction quelconque à variation totale bornée par rapport à l'ensemble de s et de t, la limite de l'expression (1) existe (§§ 8, 9) et définit une fonctionnelle bilinéaire

* Presented to the Society, Feb. 27, 1915.

† T. 150 (1910), p. 1231.

‡ *Extension au cas des intégrales multiples d'une définition de l'intégrale due à Stieltjes*, Nouvelles Annales de Mathématiques, 4e série, t. 10 (1910).

qui peut être également représentée par deux intégrales simples de Stieltjes successives (§§ 10, 11, 12). Il aurait été plus rapide de poser à priori la définition de l'intégrale double généralisée sous la forme (1) et d'en déduire la représentation des fonctionnelles bilinéaires. Mais j'ai préféré au risque de répéter plusieurs fois les mêmes raisonnements, montrer comment la définition des fonctionnelles bilinéaires conduit nécessairement à cette définition nouvelle de l'intégrale double.

J'applique ensuite les résultats obtenus aux fonctionnelles du second ordre (§ 13) et aux différentielles des fonctionnelles (§§ 14, 15, 16).

Rappel de quelques définitions et propriétés concernant les fonctionnelles lineaires

1. Suivant la terminologie de M. Hadamard, nous dirons qu'une *fonctionnelle U_f est définie* dans le champ des fonctions continues si, à toute fonction continue $f(x)$ correspond une valeur bien déterminée de U_f. (Je supposerai fixe et fini l'intervalle (a, b) où $f(x)$ est considérée.) La fonctionnelle U_f est *continue* si $U_{f_n} - U_{f_0}$ tend vers zéro lorsque la fonction continue $f_n(x)$ converge uniformément vers $f_0(x)$ dans (a, b). Elle est *distributive* lorsqu'on a l'identité $U_{f_1} + U_{f_2} = U_{f_1+f_2}$ quelles que soient les fonctions $f_1(x)$, $f_2(x)$ continues dans (a, b).

M. Hadamard appelle *fonctionnelles linéaires* les fonctionnelles qui sont distributives et continues. Elles vérifient, comme on le voit facilement, la relation

$$U_{cf(x)} = cU_{f(x)}$$

où c est un nombre quelconque. (Dans tout cet article je n'envisage que des quantités réelles). M. F. Riesz a démontré* que *toute fonctionnelle linéaire U_f peut s'écrire sous la forme*

$$U_f = \int_a^b f(s)\,du(s) \tag{1 bis}$$

où $u(s)$ est une certaine fonction à variation bornée dans (a, b), indépendante de f, et où l'intégrale est prise *au sens de Stieltjes, c'est à dire* est égale à la limite unique de la somme

$$\sum_1^n f(\sigma_i)[u(s_i) - u(s_{i-1})] \tag{2}$$

où les nombres σ_i, s_i sont des nombres quelconques tels que

$$a = s_0 \leqq \sigma_1 \leqq s_1 \leqq \sigma_2 \cdots \leqq \sigma_n \leqq s_n = b,$$

lorsque la plus grande des différences $(s_i - s_{i-1})$ tend vers zéro. M. F. Riesz a en outre montré† que si la détermination de $u(s)$ n'est pas unique,

* F. Riesz, Annales Scientifiques de l'Ecole Normale Supérieure, t. 28 (1911), p. 43.

† Loc. cit., p. 38.

du moins on ne peut la remplacer que par une fonction qui est égale à $u(s)$ augmentée d'une constante sauf peut être pour un ensemble dénombrable de valeurs de s autres que a et que b. J'ai moi-même indiqué qu'on obtient l'unicité de $u(s)$ connaissant U_f si l'on impose à $u(s)$,—ce qui est toujours possible,—la condition d'être égale à o pour $s = b$ et d'être partout régulière $[2u(s) \equiv u(s+o) + u(s-o)]$.

Nous utiliserons aussi cette propriété de U_f: si la fonction $u(s)$ est partout régulière, sa variation totale dans (a, b) est égale à la *borne réduite* de U_f,—c'est à dire à la borne supérieure de U_f/mf en désignant par mf le maximum de $|f(x)|$ dans (a, b),—quand $f(x)$ est une fonction quelconque continue dans (a, b).* Cette propriété ressort du mode de démontration de la formule (1 bis).

2. Enfin, nous aurons besoin d'une autre propriété due à F. Riesz.†

Si, quelle que soit la fonction $f(x)$ continue dans a, b, on a

$$\int_a^b f(s)\,du(s) = \lim_{n=\infty} \int_a^b f(s)\,du_n(s)$$

la variation totale de $u_n(x)$ est bornée supérieurement quand $u_n(x)$ est une fonction régulière à variation bornée. Cette propriété ayant été énoncée sans démonstration, nous allons l'établir. Il nous suffira pour cela,—suivant une indication qu'a bien voulu me communiquer M. F. Riesz,—de modifier légèrement le raisonnement employé par M. H. Lebesgue‡ pour prouver une proposition analogue.

Si le théorème n'était pas exact, on pourrait, étant donné un nombre positif quelconque l_0, trouver un entier n_0, tel que la variation totale de $\psi_{n_0}(s)$ de a à b soit supérieure à l_0, en posant $\psi_n(s) = u_n(s) - u(s)$. Mais cette variation étant la borne réduite de $\int_a^b f d\psi_{n_0}$, on pourra trouver une fonction continue f_0 telle que

$$\left| \int_a^b f_0\, d\psi_{n_0} \right| > l_0 \cdot mf_0.$$

Et d'autre part, d'après l'hypothèse, on pourra trouver un entier μ_0 tel que

$$\left| \int_a^b f_0\, d\psi_n \right| < 1$$

pour $n > \mu_0$.

En poursuivant, on pourra, d'une facon générale ayant défini les nombres $n_0, n_1, \cdots n_{p-1}, \mu_0, \cdots \mu_{p-1}, l_0, \cdots l_{p-1}$, et les fonctions continues $f_0, f_1,$

* F. Riesz, loc. cit., p. 43.

† F. Riesz, Comptes Rendus, t. 149 (1909), p. 9.

‡ H. Lebesgue, Annales de la Faculté des Sciences de Toulouse, 3e série, t. 2 (1910), p. 61.

$\cdots f_{p-1}$, poser

$$l_p = 2^p V\psi_{n_{p-1}}$$

(en appelant $V\psi$ la variation totale de ψ de a à b), trouver un entier $n_p > \mu_{p-1}$ tel que $V\psi_{n_p} > l_p$, trouver une fonction continue de a à b, f_p telle que

$$\left|\int_a^b f_p\, d\psi_{n_p}\right| > l_p \cdot mf_p,$$

et enfin trouver un entier μ_p tel que

$$\left|\int_a^b F_p\, d\psi_n\right| < 1$$

pour $n > \mu_p$, F_p étant la fonction continue

$$F_p = f_0 + k_1 f_1 + k_2 f_2 + \cdots + k_p f_p$$

où

$$k_p = \frac{1}{2^{p-1} \cdot V\psi_{n_{p-1}} \cdot mf_p}.$$

Or la fonction continue F_p converge uniformément vers une fonction limite F. Car

$$m(k_p f_p + \cdots + k_{p+q} f_{p+q}) \leqq k_p\, mf_p + \cdots + k_{p+q}\, mf_{p+q}$$

$$= \frac{1}{2^{p-1} \cdot V\psi_{n_{p-1}}} + \cdots + \frac{1}{2^{p+q-1} \cdot V\psi_{n_{p+q-1}}} < \frac{1}{l_0 \cdot 2^p}.$$

La fonction F est donc continue et on doit avoir

(2 bis) $$\lim_{n=\infty} \int_a^b F d\psi_n = 0.$$

Or

$$\left|\int_a^b F d\psi_{n_p}\right| \geqq \left|\int_a^b k_p f_p\, d\psi_{n_p}\right| - \left|\int_a^b F_{p-1}\, d\psi_{n_p}\right| - \sum_{i=p+1}^{i=\infty} \left|\int_a^b k_i f_i\, d\psi_{n_p}\right|.$$

Mais

$$\left|\int_a^b k_p f_p\, d\psi_{n_p}\right| > k_p \cdot l_p \cdot mf_p = 2; \qquad \int_a^b F_{p-1}\, d\psi_{n_p} < 1;$$

et, pour $i > p$

$$\left|\int_a^b k_i f_i\, d\psi_{n_p}\right| \leqq k_i \cdot mf_i\, V\psi_{n_p} = \frac{1}{2^{i-1}} \frac{V\psi_{n_p}}{V\psi_{i-1}} < \frac{1}{2^{i-1}}.$$

Donc on aurait quel que soit p

$$\left|\int_a^b F d\psi_{n_p}\right| \geqq \tfrac{1}{2}$$

contrairement à l'égalité (2 bis).

FONCTIONNELLES BILINÉAIRES

3. Nous nous proposons maintenant d'étudier les fonctionnelles bilinéaires, c'est à dire les fonctionnelles $U_{f,\phi}$ qui dépendent de deux fonctions continues $f(x)$, $\phi(y)$, définies dans deux intervalles fixes $a \leqq x \leqq a'$; $b \leqq y \leqq b'$, ces fonctionnelles étant linéaires séparément par rapport à $f(x)$ et par rapport à $\phi(y)$.

Nous démontrerons d'abord une propriété fondamentale de ces fonctionnelles. Par hypothèse $U_{f,\phi}$ est continue par rapport à $f(x)$ quand on y considère $\phi(x)$ comme bien déterminée et inversement. Mais est-il certain que U_{f_n,ϕ_n} converge vers $U_{f,\phi}$ lorsque f_n et ϕ_n convergent uniformément *et simultanément* vers f et ϕ? Nous allons montrer que la réponse est affirmative.

Dans ce but, remarquons que pour un choix déterminé de ϕ, on peut représenter $U_{f,\phi}$ sous la forme (1 bis) d'une intégrale de Stieltjes. Si on suppose que dans cette formule (1), $u(s)$ est nulle pour $s = b$ et qu'elle est partout régulière, $u(s)$ est bien déterminée. C'est donc une fonctionnelle de ϕ; on pourra rappeler cette dépendance en représentant $u(s)$ par la notation $v_\phi(s)$. On a donc

$$U_{f,\phi} = \int_a^{a'} f(s)\, dv_\phi(s) \tag{3}$$

et de même

$$U_{f,\phi} = \int_b^{b'} \phi(t)\, dw_f(t). \tag{3 bis}$$

On voit facilement que v_ϕ est une fonctionnelle distributive par rapport à ϕ. On peut même obtenir aussi facilement un résultat un peu plus général: soient c_1, c_2 deux nombres rationnels quelconques, ϕ_1, ϕ_2 deux fonctions continues dans (b, b'). On a

$$\int_a^{a'} f(s)\, dv_{c_1\phi_1+c_2\phi_2}(s) = U_{f,c_1\phi_1+c_2\phi_2} = c_1 U_{f\phi_1} + c_2 U_{f\phi_2}$$
$$= \int_a^{a'} f(s)\, d[c_1 v_{\phi_1}(s) + c_2 v_{\phi_2}(s)].$$

La fonction de s, $c_1 v_{\phi_1}(s) + c_2 v_{\phi_2}(s)$, est aussi nulle pour $s = b$ et elle est partout régulière, elle doit donc être égale à $v_{c_1\phi_1+c_2\phi_2}$, puisque les fonctionnelles représentées par les termes extrêmes sont identiques. On a donc

$$v_{c_1\phi_1+c_2\phi_2}(s) = c_1 v_{\phi_1}(s) + c_2 v_{\phi_2}(s). \tag{4}$$

4. Cette remarque étant faite, nous allons montrer que l'expression

$$\frac{|U_{f,\phi}|}{mf \cdot m\phi} *$$

* Je rappelle que la notation $m\phi$ désigne le maximum de la fonction continue $|\phi(s)|$ dans son intervalle de définition (b, b').

a une borne supérieure finie quand f et ϕ sont des fonctions continues quelconques dans (a, a'), (b, b'). On sait que

$$\frac{U_{f,\phi}}{mf \cdot m\phi} \leqq \frac{(\text{variation totale de } v_\phi(s) \text{ dans } (a, a'))}{m\phi}.$$

Il suffit de montrer que le second membre est borné supérieurement. Or dans le cas contraire, on pourrait trouver des fonctions ϕ_n continues dans (b, b') telles que la variation totale de v_{ϕ_n} dans $(a, a') \geqq n \cdot m\phi_n$. Posons

$$\psi_n = \frac{\phi_n}{\sqrt{n} \cdot m\phi_n},$$

on a d'après (4)

$$v_{\psi_n} = \frac{1}{\sqrt{n} \cdot m\phi_n} \cdot v_{\phi_n}$$

d'où

$$(\text{variation totale de } v_{\psi_n}) = \frac{1}{\sqrt{n} \cdot m\phi_n} (\text{variation totale de } v_{\phi_n}) \geqq \sqrt{n}.$$

Donc la variation totale de v_{ψ_n} croît indéfiniment avec n. Or il est évident que $m\psi_n = 1/\sqrt{n}$ et par suite que

$$\int_b^{b'} \psi_n(t)\, dw_f(t)$$

tend vers zéro avec $1/n$; c'est à dire que l'opération linéaire en f

$$\int_a^{a'} f(s)\, dv_{\psi_n}(s)$$

converge vers zéro avec $1/n$. Mais, d'après F. Riesz, comme nous l'avons rappelé, l'une des conditions pour qu'il en soit ainsi est que la variation totale de v_{ψ_n} soit bornée quand n croît, contrairement à l'hypothèse.

Soit donc M la borne supérieure finie du rapport

$$\frac{(\text{variation totale de } v_\phi)}{m\phi}.$$

On voit alors que

$$|U_{f_n,\phi_n} - U_{f,\phi}| \leqq |U_{f_n,\phi_n} - U_{f_n,\phi}| + |U_{f_n,\phi} - U_{f,\phi}|$$
$$\leqq M\{mf_n \cdot m(\phi_n - \phi) + m(f - f_n) \cdot m\phi\}.$$

Si f_n et ϕ_n convergent uniformément et simultanément vers f, ϕ, on voit bien maintenant que U_{f_n,ϕ_n} tend vers $U_{f,\phi}$.

Représentation des fonctionnelles bilinéaires au moyen d'intégrations successives

5. Il résulte en particulier des considérations précédentes que l'on a quelque soit s

$$|v_\phi(s) - v_\phi(a')| \leqq (\text{variation totale de } v_\phi \text{ dans } (a, a')) \leqq M \cdot m\phi$$

et puisque $v_\phi(a') = 0$ par hypothèse,

$$|v_\phi(s)| \leqq M \cdot m\phi \tag{6}$$

ou

$$|v_{\phi_1 - \phi_2}| \leqq M \cdot m(\phi_1 - \phi_2). \tag{7}$$

D'après (4) et (7), v_ϕ est donc une fonctionnelle linéaire de ϕ et l'on peut écrire

$$v_\phi(s) = \int_b^{b'} \phi(t)\, du(s,t), \tag{8}$$

$u(s,t)$ étant une fonction à variation bornée en t, nulle pour $t = b'$ et régulière partout en t. D'ailleurs $v_\phi(a') = 0$ quelque soit ϕ; on doit donc prendre $u(a,t) \equiv 0$. De sorte qu'on *peut finalement représenter toute fonctionnelle bilinéaire sous la forme*

$$U_{f,\phi} = \int_a^{a'} f(s) \cdot d_s \left[\int_b^{b'} \phi(t)\, d_t\, u(s,t) \right] \tag{9}$$

ou aussi, en raisonnant de même à partir de la formule (3 bis),

$$U_{f,\phi} = \int_b^{b'} \phi(t)\, d_t \left[\int_a^{a'} f(s)\, d_s\, u_1(s,t) \right]. \tag{9 bis}$$

Dans ces expressions, l'indice de d_s et d_t indique si l'on doit, dans le terme qui suit, faire varier s ou t. Les deux fonctions $u(s,t)$, $u_1(s,t)$ sont bien définies pour $a \leqq s \leqq a'$; $b \leqq t \leqq b'$, nulles toutes deux pour $s = a'$ et pour $t = b'$, la première régulière et à variation bornée par rapport à t, la deuxième régulière et à variation bornée par rapport à s.

Représentation de toute fonctionnelle bilinéaire par une intégrale double

6. Nous allons utiliser une formule analogue à la formule de la moyenne et très simple à établir,

$$\left| \int_a^{a'} f(s)\, d\alpha(s) - \sum_1^n f(\sigma_i)[\alpha(s_i) - \alpha(s_{i-1})] \right| \leqq w_{f,s} \cdot (\text{variation totale de } \alpha), \tag{10}$$

où $a = s_0 \leqq \sigma_1 \leqq s_1 \leqq \cdots \leqq \sigma_n \leqq s_n = a'$, et où $w_{f,s}$ est l'oscillation

maximum de $f(s)$ dans un intervalle de longueur ϵ égal au plus grand des intervalles (s_{i-1}, s_i). En prenant pour $\alpha(s)$ la fonction $v_\phi(s)$, on a donc

$$(10\text{ bis}) \quad |U_{f,\phi} - \sum_1^n f(\sigma_i)[v_\phi(s_i) - v_\phi(s_{i-1})]| \leqq M \cdot m\phi \cdot w_{f,\epsilon}.$$

Mais on a d'après (8)

$$(11) \quad \sum_1^n f(\sigma_i)[v_\phi(s_i) - v_\phi(s_{i-1})] = \int_b^{b'} \phi(t)\, d\lambda_f(t),$$

où $\lambda_f(t)$ est la fonction régulière et à variation bornée, nulle pour $t = b'$,

$$(12) \quad \lambda_f(t) = \sum_1^n f(\sigma_i)[u(s_i, t) - u(s_{i-1}, t)].$$

Or

$$(13) \quad \left| \int_b^{b'} \phi(t)\, d\lambda_f - \sum_{j=1}^{j=p} \phi(\tau_j)[\lambda_f(t_j) - \lambda_f(t_{j-1})] \right| \leqq w_{\phi,\epsilon'} \cdot (\text{variation totale de } \lambda_f(t)),$$

où $b = t_0 \leqq \tau_1 \leqq t_1 \leqq \tau_2 \cdots \leqq \tau_p \leqq t_p = b'$, et où ϵ' est la plus grande longueur des intervalles (t_{j-1}, t_j). Or d'après (11)

$$\left| \int_b^{b'} \phi(t)\, d\lambda_f(t) \right| \leqq mf \cdot (\text{variation totale de } v_\phi(s)) \leqq M \cdot mf \cdot m\phi$$

et comme la variation totale de λ_f est égale à la borne réduite de la fonctionnelle linéaire en ϕ, $\int_b^{b'} \phi(t)\, d\lambda_f(t)$, on a

$$(14) \quad \text{variation totale de } \lambda_f \leqq M \cdot mf.$$

En combinant (10 bis), (11), (13), (14), on obtient l'inégalité

$$\left| U_{f,\phi} - \sum_{j=1}^{j=p} \phi(\tau_j)[\lambda_f(t_j) - \lambda_f(t_{j-1})] \right| \leqq M \cdot [m\phi \cdot w_{f,\epsilon} + mf \cdot w_{\phi,\epsilon'}]$$

ou d'après (12)

$$(15) \quad \left| U_{f,\phi} - \sum_{j=1}^{j=p} \sum_{i=1}^{i=n} \phi(\tau_j) f(\sigma_i)[u(s_i, t_j) - u(s_{i-1}, t_j) - u(s_i, t_{j-1}) + u(s_{i-1}, t_{j-1})] \right| \leqq M[m\phi \cdot w_{f,\epsilon} + mf \cdot w_{\phi,\epsilon'}].$$

Quand on fait tendre ϵ et ϵ' simultanément vers zéro, $w_{f,\epsilon}$ et $w_{\phi,\epsilon'}$ tendent simultanément vers zéro. Par suite

$$(16) \quad U_{f,\phi} = \lim_{\substack{\epsilon=0 \\ \epsilon'=0}} \sum_{i,j} f(\sigma_i)\phi(\tau_j)\Delta_{i,j}u,$$

où $\Delta_{i,j}\, u$ est une différence seconde de u

$$\Delta_{i,j}\, u = u(s_i, t_j) - u(s_{i-1}, t_j) - u(s_i, t_{j-1}) + u(s_{i-1}, t_{j-1}).$$

Ceci nous montre d'abord qu'il existe une fonction $u(s,t)$ indépendante de f et de ϕ telle que le second membre de (16) ait une limite unique quels que soient les modes de division en s_i, t_j des intervalles (a, a'), (b, b'). Il est alors naturel de *considérer ce second membre comme représentant une généralisation de l'intégrale double et de le représenter par la notation*

$$\int_a^{a'} \int_b^{b'} f(s) \cdot \phi(t) \cdot d_s\, d_t\, u(s,t). \tag{17}$$

Et la formule (16) nous fournit alors *une nouvelle représentation des fonctionnelles bilinéaires*

$$U_{f,\phi} = \int_a^{a'} \int_b^{b'} f(s) \cdot \phi(t) \cdot d_s\, d_t\, u(s,t). \tag{18}$$

La formule (15) peut alors être considérée comme une généralisation de la formule (10).

PROPRIÉTÉ CARACTÉRISTIQUE DE LA FONCTION $u(s,t)$

7. Soient $s, \cdots, s_n, t_1, t_2, \cdots$, des nombres quelconques tels que

$$a = s_0 \leqq s_1 \leqq s_2 \cdots \leqq s_n = a'; \qquad b = t_0 \leqq t_1 \leqq t_2 \cdots \leqq t_p = b'$$

et $\epsilon_1, \cdots, \epsilon_n, \epsilon'_1, \cdots, \epsilon'_p$ des nombres égaux à 1 ou à -1, mais de signes arbitraires. Je dis que *la quantité*

$$\left| \sum_{i=1}^{i=n} \sum_{j=1}^{j=p} \epsilon_i\, \epsilon'_j\, [u(s_i, t_j) - u(s_{i-1}, t_j) - u(s_i, t_{j-1}) + u(s_{i-1}, t_{j-1})] \right| \tag{19}$$

a une borne supérieure finie quand on fait varier de façon quelconque les nombres $s_i, t_j, \epsilon_i, \epsilon'_j, n, p$ *(choisis cependant comme il vient d'être spécifié).* Et de plus *cette borne supérieure est égale à la "borne réduite" de* $U_{f,\phi}$ c'est à dire à la borne supérieure de $U_{f,\phi}/mf \cdot m\phi$ dans l'ensemble des fonctions continues f, ϕ.

On a d'abord évidemment

$$|\sum_i \epsilon_i [v_\phi(s_i) - v_\phi(s_{i-1})]| \leqq \sum_i |v_\phi(s_i) - v_\phi(s_{i-1})|$$

$$\leqq (\text{variation totale de } v_\phi) \leqq M \cdot m\phi.$$

Or

$$\sum_i \epsilon_i [v_\phi(s_i) - v_\phi(s_{i-1})] = \int_b^{b'} \phi(t)\, d\mu(t)$$

avec

$$\mu(t) = \sum_i \epsilon_i [u(s_i, t) - u(s_{i-1}, t)].$$

La fonction $\mu(t)$ est régulière et à variation bornée; sa variation totale est

donc la borne supérieure de

$$\frac{\left|\int_b^{b'} \phi(t)\,d\mu(t)\right|}{m\phi} \leqq \frac{\sum_i \epsilon_i [v_\phi(s_i) - v_\phi(s_{i-1})]}{m\phi} \leqq M.$$

On a donc

$$|\sum_j \epsilon'_j [\mu(t_j) - \mu(t_{j-1})]| \leqq (\text{variation totale de } \mu(t)) \leqq M.$$

Or le premier membre n'est autre que l'expression (19). Nous avons donc bien prouvé que cette expression reste bornée supérieurement et même que sa borne supérieure est au plus égale à celle, M, du rapport

$$\frac{(\text{variation totale de } v_\phi)}{m\phi}.$$

Celle ci est égale à la borne réduite M' de $U_{f,\phi}$; nous avons déjà vu en effet (§ 4) qu'elle lui est au moins égale: $M \geqq M'$. D'autre part quand ϕ est fixé, on peut choisir une fonction f telle que le rapport $\left|\int_a^{a'} f(s)\,dv_\phi(s)\right|/mf$ soit voisin de la variation totale de v_ϕ. Donc si on a préalablement choisi ϕ de sorte que le rapport

$$\frac{(\text{variation totale de } v_\phi)}{m\phi}$$

soit suffisamment voisin de sa borne supérieure M, l'expression $U_{f,\phi}/mf \cdot m\phi$ sera aussi voisine que l'on veut de M. Par suite $M \leqq M' \leqq M$, d'où $M = M'$. On a ainsi démontré l'inégalité

$$|\sum_{i,j} \epsilon_i \epsilon'_j \Delta_{i,j} u| \leqq M, \tag{20}$$

où M est maintenant aussi la borne réduite de $U_{f,\phi}$. Nous verrons un peu plus loin, § 10, que la borne supérieure N du premier membre non seulement, comme le démontre cette inégalité, est au plus égale à M; mais encore qu'elle lui est exactement égale.

8. Une conséquence importante de l'inégalité (20) est la suivante. Nous avons vu que $u(s,t)$ est à variation bornée par rapport à t. Nous allons voir qu'il en est de même par rapport à s et que la *variation totale de* $u(s,t)$, *soit par rapport à* s, *soit par rapport à* t, *reste inférieure ou égale à un nombre indépendant de* t *et de* s *et en particulier à* M. Appliquons en effet l'inégalité (20) en prenant $p = 2$. On aura:

$$|\epsilon'_1 \sum_i \epsilon_i \Delta_{i1} u + \epsilon'_2 \sum_i \epsilon_i \Delta_{i2} u| \leqq M.$$

Prenons les ϵ_i de sorte que les $\epsilon_i \Delta_{i2} u$ soient $\geqq 0$; puis, ayant fixé les ϵ_i, prenons

$\epsilon_2' = 1$ et ϵ_1' de sorte que $\epsilon_1' \sum_i \epsilon_i \Delta_{i1} u \geqq 0$, on aura alors

$$\sum_i |\Delta_{i2} u| \leqq |\epsilon_1' \sum_i \epsilon_i \Delta_{i1} u + \epsilon_2' \sum_i \epsilon_i \Delta_{i2} u| \leqq M.$$

Comme d'autre part $t_2 = b'$ et $u(s, b') = 0$, on aura

$$\Delta_{i2} u = u(s_i, t_1) - u(s_{i-1}, t_1);$$

d'où enfin

$$\sum_{i=1}^{i=n} |u(s_i, t_1) - u(s_{i-1}, t_1)| \leqq M.$$

Ainsi $u(s, t_1)$ est à variation bornée quel que soit le nombre arbitraire t_1 et sa variation est au plus égale à M. On verrait de même que la variation totale de $u(s_1, t)$—qu'on sait déjà être bornée quel que soit s_1—reste au plus égale à M.

Définition générale de l'intégrale double

$$\int_a^{a'} \int_b^{b'} f(s) \cdot \phi(t) \cdot d_s d_t k(s, t)$$

9. La fonctionnelle bilinéaire $U_{f,\phi}$ nous a conduit à la conception de l'intégrale double

$$\int_a^{a'} \int_b^{b'} f(s) \cdot \phi(t) \cdot d_s d_t u(s, t). \tag{21}$$

Il est naturel de se demander si, sans s'occuper de l'origine de la fonction $u(s, t)$, les propriétés de la fonction u suffisent à donner un sens à l'intégrale double. Nous allons démontrer que la réponse est affirmative.

Ainsi nous allons partir *d'une fonction $k(s, t)$ dont on sait seulement qu'elle est à variation bornée par rapport à l'ensemble de s et de t. Nous entendons par là que l'expression*

$$\left| \sum_{i=1}^{i=n} \sum_{j=1}^{j=p} \epsilon_i \epsilon_j' [k(s_i, t_j) - k(s_{i-1}, t_j) - k(s_i, t_{j-1}) + k(s_{i-1}, t_{j-1})] \right| \tag{22}$$

reste inférieure à un nombre fixe P quels que soient les nombres n, p, ϵ_i, ϵ_j', s_i, t_j tels que

$$|\epsilon_i| = |\epsilon_j'| = 1; \qquad a = s_0 \leqq s_1 \cdots \leqq s_n = a'; \qquad b = t_0 \leqq t_1 \cdots \leqq t_p = b'.$$

Et nous allons montrer que si ϵ, ϵ' désignent les plus grandes des quantités $(s_i - s_{i-1})$ et $(t_j - t_{j-1})$, si $f(s)$ et $\phi(t)$ désignent deux fonctions uniformément continues respectivement pour $a \leqq s \leqq a'$; $b \leqq t \leqq b'$, et σ_i, τ_j un point quelconque du rectangle $s_{i-1} \leqq \sigma_i \leqq s_i$; $t_{j-1} \leqq \tau_j \leqq t_j$, *l'expression*

$$S = \sum_{i=1}^{i=n} \sum_{j=1}^{j=p} f(\sigma_i) \cdot \phi(\tau_j) \Delta_{i,j} k(s, t)$$

a une limite déterminée et unique quand ϵ et ϵ' tendent simultanément vers zéro.

En effet, soit un autre système de valeurs des nombres $n, p, \sigma_i, \tau_j, s_i, t_j$; à savoir $n', p', \sigma'_{i'}, \tau'_{j'}, s'_{i'}, t'_{j'}$, pour lequel les plus grandes des quantités $s'_{i'} - s'_{i'-1}$, $t'_{j'} - t'_{j'-1}$ sont respectivement inférieures à η, η'. Combinons les deux systèmes en rangeant par ordre de grandeur les $s_i, s'_{i'}$ dans une seule suite $s''_{i''}$ et les $t_j, t'_{j'}$ en une seule suite $t''_{j''}$. Il est facile de voir que $\Delta_{i,j}$ sera la somme de plusieurs quantités $\Delta_{i'',j''}$ et de même pour $\Delta_{i',j'}$; que, par suite, on a

$$\sum_{i,j} f(\sigma_i)\phi(\tau_j)\Delta_{i,j}\,k - \sum_{i',j'} f(\sigma'_{i'})\phi(\tau'_{j'})\Delta_{i',j'}\,k = \sum_{i'',j''} [f(\sigma_i)\phi(\tau_j) - f(\sigma'_{i'})\phi(\tau'_{j'})]\,\Delta_{i'',j''}\,k, \tag{23}$$

où

$$s_{i-1} \leqq s''_{i''-1} \leqq s''_{i''} \leqq s_i; \qquad s'_{i'-1} \leqq s''_{i''-1} \leqq s''_{i''} \leqq s'_{i'}; \qquad \text{et de même pour les } t.$$

De sorte que

$$\sigma_i - \sigma'_{i'} = (\sigma_i - s_{i-1}) - (s''_{i''-1} - s_{i-1}) - (s'_{i'} - s''_{i''-1}) + (s'_{i'} - \sigma'_{i'}),$$

d'où

$$|\sigma_i - \sigma'_{i'}| \leqq 2(\epsilon + \eta),$$

et de même

$$|\tau_j - \tau'_{j'}| \leqq 2(\epsilon' + \eta').$$

Si donc $\epsilon + \eta < \theta/2$ et $\epsilon' + \eta' < \theta'/2$ on aura

$$|f(\sigma_i) - f(\sigma'_{i'})| \leqq w_{f,\theta}, \qquad |\phi(\tau_j) - \phi(\tau'_{j'})| \leqq w_{\phi,\theta'}.$$

Or, en appellant $S - S'$ la différence (23), on aura

$$|S - S'| = |\sum_{i'',j''} f(\sigma_i)[\phi(\tau_j) - \phi(\tau'_{j'})]\,\Delta_{i'',j''}\,k + \phi(\tau'_{j'})[f(\sigma_i) - f(\sigma'_{i'})]\,\Delta_{i'',j''}\,k| = mf \cdot w_{\phi,\theta'} |\sum_{i''} \Theta''_{i''}[\sum_{j''} \Omega'_{j''}\,\Delta''_{i'',j''}]| + m\phi \cdot w_{f,\theta} |\sum_{j''} \Theta'_{j''}[\sum_{i''} \Omega_{i''}\,\Delta_{i'',j''}]|,$$

où les $\Theta, \Theta', \Omega, \Omega'$ sont en valeurs absolues inférieurs ou égaux à l'unité. D'où

$$|S - S'| \leqq mf \cdot w_{\phi,\theta'} \sum_{i''} |\sum_{j''} \Omega'_{j''}\,\Delta_{i'',j''}| + m\phi \cdot w_{f,\theta} \sum_{j''} |\sum_{i''} \Omega_{i''}\,\Delta_{i'',j''}|. \tag{24}$$

On peut choisir des nombres $\epsilon_{i''} = \pm 1$ tels que

$$\sum_{i''} |\sum_{j''} \Omega'_{j''}\,\Delta_{i'',j''}| = |\sum_{i''} \epsilon_{i''} \sum_{j''} \Omega'_{j''}\,\Delta_{i'',j''}| = |\sum_{j''} \Omega'_{j''} \sum_{i''} \epsilon_{i''}\,\Delta_{i'',j''}| \leqq \sum_{j''} |\sum_{i''} \epsilon_{i''}\,\Delta_{i'',j''}|.$$

En choisissant convenablement des nombres $\epsilon'_{j''} = \pm 1$, on aura

$$\epsilon'_{j''} \sum_{i''} \epsilon_{i''} \Delta_{i'',j''} = \left| \sum_{i''} \epsilon_{i''} \Delta_{i'',j''} \right|,$$

d'où

$$\sum_{i''} \left| \sum_{j''} \Omega'_{j''} \Delta_{i'',j''} \right| = \left| \sum_{i''} \sum_{j''} \epsilon_{i''} \epsilon'_{j''} \Delta_{i'',j''} \right| \leqq P.$$

En opérant de même sur le second terme de l'inégalité (24) on arrive finalement à la formule

(24 bis) $$|S - S'| \leqq P\,[mf \cdot w_{\phi,\theta'} + m\phi \cdot w_{f,\theta}].$$

On en déduit par un raisonnement classique, que si ϵ, ϵ', η, η' tendent vers zéro simultanément, S et S' ont une limite qui est lø même. Ainsi, quand ϵ, ϵ' tendent vers zéro simultanément S a une limite déterminée, indépendante du choix des nombres n, p, s_i, t_j, σ_i, τ_j; limite que nous désignerons par

$$\int_a^{a'} \int_b^{b'} f(s)\,\phi(\tau)\,d_s\,d_t\,k(s,t).$$

De plus, si on laisse ϵ, ϵ' fixes et si on prend les points de division $s'_{i'}$, $t'_{j'}$, de sorte qu'ils comprennent parmi eux tous les points s_i, t_j on aura $|\sigma_i - \sigma'_{i'}| < \epsilon$, $|\tau_i - \tau'_{i'}| < \epsilon'$; on pourra donc, dans la formule (24 bis), remplacer θ, θ' par ϵ, ϵ'. Si maintenant on fait tendre η, η' vers zéro en laissant ϵ, ϵ' fixes, la formule (24 bis) devient

(25) $$\left| \int_a^{a'} \int_b^{b'} f(s)\,\phi(t)\,d_s\,d_t\,k(s,t) - S \right| \leqq P\,(mf \cdot w_{\phi,\epsilon'} + m\phi \cdot w_{f,\epsilon}).$$

L'INTÉGRALE DOUBLE $\int_a^{a'} \int_b^{b'} f(s) \cdot \phi(t)\,d_s\,d_t\,k(s,t)$ EST UNE FONCTIONNELLE BILINÉAIRE

10. Nous arrivons maintenant à la réciproque du théorème concernant la représentation d'une fonctionnelle bilinéaire par l'intégrale double généralisée (17). Posons

$$K_{f,\phi} = \int_a^{a'} \int_b^{b'} f(s) \cdot \phi(t)\,d_s\,d_t\,k(s,t).$$

Il est manifeste, d'après l'expression de S que sa limite $K_{f,\phi}$ est une fonctionnelle distributive par rapport à f et à ϕ séparément. Pour montrer que c'est une fonctionnelle bilinéaire en f et ϕ, il suffit (§ 4) de prouver que $K_{f,\phi}/mf \cdot m\phi$ est borné supérieurement. Or on a

$$\frac{|K_{f,\phi}|}{mf \cdot m\phi} = \lim \frac{|S|}{mf \cdot m\phi} = \lim \left| \sum_i \sum_j \frac{f(\sigma_i)}{mf} \cdot \frac{\phi(\tau_j)}{m\phi} \cdot \Delta_{i,j}k \right|.$$

Mais un raisonnement développé précédemment (§ 9) nous montrait que

l'on a

$$|\sum_i \sum_j \Theta_i \Omega_j \Delta_{i,j} k| \leqq P,$$

sous la seule condition que les Θ_i et Ω_j soient en valeurs absolues au plus égaux à 1. On a donc

$$\left|\sum_i \sum_j \frac{f(\sigma_i)}{mf} \cdot \frac{\phi(\tau_j)}{m\phi} \cdot \Delta_{i,j} k\right| \leqq P$$

et par suite $K_{f,\phi}$ a une "borne réduite" finie et celle-ci est au plus égale à P. Il en résulte immédiatement que $K_{f,\phi}$ est continue par rapport à f et ϕ et par suite bilinéaire.

Ainsi *quelle que soit la fonction $k(s,t)$ à variation bornée par rapport à l'ensemble de s et t,* l'intégrale double*

$$\int_a^{a'} \int_b^{b'} f(s) \cdot \phi(t) \cdot d_s\, d_t\, k(s,t)$$

a un sens et représente une fonctionnelle bilinéaire $K_{f,\phi}$. De plus *la "borne réduite" de K*, c'est-à-dire la borne supérieure de $|K_{f,\phi}|/mf \cdot m\phi$ *est au plus égale à la variation totale de $k(s,t)$*, c'est à dire à la borne supérieure de l'expression (22). Enfin, on peut, sans changer la valeur de la fonctionnelle, remplacer $k(s,t)$ par une certaine fonction $u(s,t)$ également à variation bornée par rapport à l'ensemble de s et de t et dont la variation totale est exactement égale à la borne réduite de $K_{f,\phi}$. Ce dernier point résulte de la combinaison des résultats de ce numéro et du numéro 7.

Représentation de l'intégrale double par deux intégrales successives

11. Il est manifeste que si l'on sait seulement que $k(s,t)$ est à variation bornée par rapport à l'ensemble de s et de t, on n'a plus le droit de calculer cette intégrale double comme on l'a fait pour la fonction particulière u au moyen de deux intégrations successives sous la forme

$$(26) \qquad \int_a^{a'} \int_b^{b'} f(s) \cdot \phi(t)\, d_s\, d_t\, k(s,t) = \int_a^{a'} f(s)\, d_s \left\{ \int_b^{b'} \phi(t)\, d_t\, k(s,t) \right\}.$$

En effet, il faudrait au moins pour cela que l'accolade ait un sens; or la fonction $k(s,t)$ n'est pas nécessairement à variation bornée par rapport à l'une des variables s, t considérée isolément. Si, par exemple, on désigne par $k(t)$ une fonction à variation non bornée dans (b, b'), on a le droit de prendre $k(s,t) = k(t)$ puisque cette fonction est à variation totale évidemment

* Je rappelle que la variation totale de $k(s,t)$ n'est pas la borne supérieure de $|\sum_i \sum_j \Delta_{i,j} k|$, mais celle de $|\sum_i \sum_j \epsilon_i \epsilon_j' \Delta_{i,j} k|$ comme il a été précisé plus haut, §9.

nulle par rapport à l'ensemble de s et de t. Dès lors pour que la relation (26) ait un sens, il faut au moins supposer que $k(s,t)$, fonction à variation bornée par rapport à l'ensemble de s et t, soit aussi à variation bornée par rapport à t.

Remarquons en passant que si $k(s,t)$, à variation bornée par rapport à l'ensemble de s et t, est à variation bornée pour *une* valeur particulière s' de s, elle est à variation bornée pour *toute* valeur de s. Posons en effet $k_1(s,t) = k(s,t) - k(s,b')$. On aura évidemment $\Delta_{i,j} k_1 = \Delta_{i,j} k$. Donc k_1 sera aussi à variation bornée par rapport à l'ensemble de s et t et sera de plus nulle pour $t = b'$. Cela suffit pour qu'on puisse appliquer un raisonnement analogue à celui du numéro 8 et par suite pour que $k_1(s,t)$ soit à variation bornée en t quelque soit s. Alors

$$k(s,t) = k(s',t) + k_1(s,t) - [k_1(s',t) + k(s',b') - k(s,b')]$$

est la somme de trois fonctions à variation bornée en t quelque soit s; elle jouit donc de la même propriété. Nous allons montrer que cette condition nécessaire pour l'écriture de (26) est aussi suffisante pour que cette égalité (26) ait un sens et soit exacte. En effet $k(s,t)$ étant à variation bornée en t quel que soit s, l'intégrale

$$\gamma_\phi(s) = \int_b^{b'} \phi(t)\, d_t\, k(s,t)$$

a un sens et représente une fonctionnelle linéaire par rapport à ϕ, bien définie pour toute valeur de s. C'est en même temps une fonction de s à variation bornée. En effet quelle que soit la suite $a = s_0 \leqq s_1 \cdots \leqq s_n = a'$, on a en choisissant convenablement les ϵ

$$\sum_i |\gamma_\phi(s_i) - \gamma_\phi(s_{i-1})| = \Sigma \epsilon_i\,]\gamma_\phi(s_i) - \gamma_\phi(s_{i-1})] = \int_b^{b'} \phi(t)\, d_t\, l(t),$$

où

$$l(t) = \sum_i \epsilon_i [k(s_i,t) - k(s_{i-1},t)]$$

est une fonction à variation bornée. Donc

$$\sum_i |\gamma_\phi(s_i) - \gamma_\phi(s_{i-1})| \leqq m\phi \cdot (\text{variation totale de } l(t)).$$

Or en prenant arbitrairement la suite $b = t_0 \leqq t_1 \cdots \leqq t_p = b'$, et choisissant convenablement les ϵ'_j, on a

$$\sum_j |l(t_j) - l(t_{j-1})| = |\sum_j \epsilon'_j [l(t_j) - l(t_{j-1})]| = |\sum_i \sum_j \epsilon_i\, \epsilon'_j\, \Delta_{i,j} k(s,t)| \leqq P$$

où P est la variation totale de $k(s,t)$ par rapport à l'ensemble de s et de t. Donc enfin, quels que soient les s_i,

$$\sum_i |\gamma_\phi(s_i) - \gamma_\phi(s_{i-1})| \leqq P \cdot m\phi.$$

On en conclut d'abord que $\gamma_\phi(s)$ est une fonction de s à variation bornée. Par conséquent l'intégrale

$$H_{f,\phi} = \int_a^{a'} f(s)\, d_s\, \gamma_\phi(s) = \int_a^{a'} f(s)\, d_s \int_b^{b'} \phi(t)\, d_t\, k(s,t)$$

a un sens.

Il reste à prouver que $K_{f,\phi} = H_{f,\phi}$. Il suffit d'utiliser dans ce but l'inégalité (25) en y laissant d'abord ϵ fixe et faisant tendre ϵ' vers zéro. Le second membre tendra vers $P \cdot m\phi \cdot w_{f,\epsilon}$; d'autre part la quantité

$$S = \sum_i f(\sigma_i)\{\sum_j \phi(\tau_j)[k(s_i,t_j) - k(s_i,t_{j-1})] - \sum_j \phi(\tau_j)[k(s_{i-1},t_j) - k(s_{i-1},t_{j-1})]\}$$

tendra évidemment vers

$$\sum_i f(\sigma_i)\left\{\int_b^{b'} \phi(t)\, dk(s_i,t) - \int_b^{b'} \phi(t)\, dk(s_{i-1},t)\right\} = \sum_i f(\sigma_i)[\gamma_\phi(s_i) - \gamma_\phi(s_{i-1})].$$

On a donc

$$|K_{f,\phi} - \sum_i f(\sigma_i)[\gamma_\phi(s_i) - \gamma_\phi(s_{i-1})]| \leqq P \cdot m\phi \cdot w_{f,\epsilon}.$$

Si dans cette égalité on fait maintenant tendre ϵ vers zéro, on obtiendra à la limite l'égalité annoncée $H_{f,\phi} = K_{f,\phi}$.

12. Nous venons de prouver qu'on peut calculer l'intégrale double $K_{f,\phi}$ par deux intégrations successives *sous la forme* (26) quand $k(s,t)$ est non seulement à variation bornée par rapport à l'ensemble de s et t mais encore à variation bornée par rapport à t pour au moins une valeur de s. Il faut bien remarquer que *si cette dernière condition n'est pas remplie, on pourra encore calculer l'intégrale double par deux intégrations successives,* mais celles-ci se présenteront sous une forme un peu plus compliquée que dans (26). Il suffit en effet de remarquer que la fonction $k_0(s,t) = k(s,t) - k(s,b')$ vérifie les deux conditions exigées pour $k(s,t)$ dans (26). On a donc

$$\int_a^{a'} \int_b^{b'} f(s) \cdot \phi(t)\, d_s\, d_t\, k_0(s,t) = \int_a^{a'} f(s)\, d_s \int_b^{b'} \phi(t)\, d_t\, k_0(s,t).$$

Or $\Delta_{i,j} k_0 = \Delta_{i,j} k$. Donc le premier membre est exactement égal à $K_{f,\phi}$. On peut raisonner de même en permutant s et t. En définitive *sous la seule condition que $k(s,t)$ ait une variation bornée par rapport à l'ensemble de s et t,* on aura

$$\int_a^{a'} \int_b^{b'} f(s) \cdot \phi(t)\, d_s\, d_t\, k(s,t) = \int_a^{a'} f(s)\, d_s \left\{\int_b^{b'} \phi(t)\, d_t [k(s,t) - k(s,b')]\right\} = \int_b^{b'} \phi(t)\, d_t \left\{\int_a^{a'} f(s)\, d_s [k(s,t) - k(s',t)]\right\}.$$

FONCTIONNELLES DU SECOND ORDRE

13. Suivant la définition générale que j'ai donnée* des fonctionnelles d'ordre n, j'appellerai fonctionnelle du second ordre une fonctionnelle continue U_f telle que l'on ait identiquement

$$U_{f_1+f_2+f_3} - U_{f_1+f_2} - U_{f_2+f_3} - U_{f_3+f_1} + U_{f_1} + U_{f_2} + U_{f_3} - U_0 \equiv 0$$

quelles que soient les fonctions f_1, f_2, f_3 continues dans un intervalle fixe (a, b).

Une telle fonctionnelle est la somme† de trois fonctionnelles dont l'une se réduit à une constante, la seconde à une fonctionnelle linéaire, la troisième à une fonctionnelle du second ordre V_f *homogène*, c'est à dire telle que

$$V_{cf} = c^2 V_f$$

quelle que soit la constante réelle c. Il suffit donc de se borner à l'étude de cette dernière. On en trouvera facilement l'expression en ramenant au calcul d'une fonctionnelle bilinéaire. Posons en effet

$$W_{f_1, f_2} = V_{f_1+f_2} - V_{f_1} - V_{f_2} + V_0.$$

On aura

$$W_{f_1+f_3, f_2} - W_{f_1, f_2} - W_{f_3, f_2} = V_{f_1+f_2+f_3} - V_{f_1+f_2} - V_{f_2+f_3} - V_{f_3+f_1} + V_{f_1} + V_{f_2} + V_{f_3} - V_0.$$

Or le second membre est identiquement nul par hypothèse. Donc W_{f_1, f_2} est distributive en f_1; de même elle est distributive en f_2. Etant continue en f_1 et en f_2, c'est une fonctionnelle bilinéaire qui peut donc être mise sous la forme

$$W_{f_1, f_2} = \int_a^b f_1(s) f_2(t)\, d_s\, d_t\, w(s, t).$$

Or, d'après la condition d'homogénéité de V, on a

$$V_0 = 0, \qquad V_{2f} = 4V_f.$$

On a donc $W_{2f} = 2V_f$. D'où, en posant $v(s, t) = \frac{1}{2} w(s, t)$,

$$V_f = \int_a^b \int_a^b f(s) f(t)\, d_s\, d_t\, v(s, t). \tag{27}$$

Telle est l'expression générale des fonctionnelles homogènes du second ordre. Dans cette expression, permutons s et t, puis faisons la demi-somme des résultats obtenus, on aura

$$V_f = \int_a^b \int_a^b f(s) f(t)\, d_s\, d_t\, v_1(s, t),$$

* *Sur les fonctionnelles continues*, Annales Scientifiques de l'Ecole Normale Supérieure, t. 27 (1910), p. 204.

† Loc. cit.

où $v_1(s, t)$ est la demi-somme de $v(s, t)$ et de $v(t, s)$, c'est à dire où v_1 est une fonction symétrique. Ainsi dans la formule (27), la fonction $v(s, t)$ est une fonction à variation bornée par rapport à l'ensemble de s et de t et *l'on peut supposer que c'est une fonction symétrique de s et de t*. Il est bien évident que réciproquement, l'expression (27) définit une fonctionnelle du second ordre homogène.

Application aux differentielles du second ordre des fonctionnelles

14. Nous avons déjà indiqué ailleurs* les avantages que présente la définition proposée par Stolz pour la différentielle totale. Outre la supériorité qu'elle possède sur la définition ordinaire dans le cas des fonctions de plusieurs variables, celui que considérait Stolz,—supériorité qui a été mise en évidenee pour la première fois par M. W. H. Young† et qui a récemment déterminé M. de la Vallée Poussin à l'adopter systématiquement dans la dernière édition de son *Cours d'Analyse*,—cette définition se prête à une généralisation immédiate dans le *Calcul Fonctionnel*. C'est même en partant de ce dernier point de vue que j'avais retrouvé la définition de Stolz.

Sans insister davantage,‡ je partirai de la définition suivante: une fonctionnelle U_f admet une différentielle pour $f = f_0$, s'il existe une fonctionnelle linéaire par rapport à l'accroissement Δf de f et que je désignerai par la notation $\partial_{\Delta f} U_{f_0}$, qui ne diffère de l'accroissement correspondant de U_f que par une quantité infiniment petite par rapport à l'écart des arguments f_0 et $f_0 + \Delta f$.

Dans le cas actuel où les fonctions $f(x)$ sont supposées continues dans un intervalle fixe (a, b) et la limite considérée pour ces fonctions est la limite uniforme, nous appellerons écart de f_0 et de $f_0+\Delta f$, le maximum de $|\Delta f|$ dans (a, b) et nous le représenterons par $m\Delta f$. On aura ainsi

$$\frac{(U_{f_0+\Delta f} - U_{f_0}) - \partial_{\Delta f} U_{f_0}}{m\Delta f} = \epsilon,$$

ϵ étant une quantité qui tend vers zéro en même temps que $m\Delta f$.

15. La définition de la différentielle seconde est alors toute naturelle. Nous dirons qu'une fonctionnelle U_f admet une différentielle seconde pour $f = f_0$ si 1° elle admet une différentielle première dans le voisinage de f_0, soit $\partial_{\Delta f} U_f$; 2° cette différentielle première admet elle même une différentielle première par rapport à un nouvel accroissement $\Delta' f$, pour $f = f_0$ et quel

* Voir pour plus de détails sur cette question ma note: *Sur la notion de différentielle totale*, Nouvelles Annales de Mathématiques, 4e série, t. 12 (1912), pp. 385–433.

† W. H. Young, *The fundamental theorems of differential calculus.*

‡ Je me suis expliqué à ce sujet dans une communication *Sur la notion de differentielle dans le calcul fonctionnel*, Comptes Rendus du Congrès des Sociétés Savantes, 1912.

que soit Δf. Nous désignerons cette différentielle seconde par $\partial_{\Delta' f}\,\partial_{\Delta f}\,U_{f_0}$ et nous démontrerons immédiatement que *cette différentielle seconde est symétrique par rapport aux deux accroissements* $\Delta' f$, Δf.*

Nous supposons que U_f admet une différentielle pour

$$m(f - f_0) < \eta,$$

où η est un nombre convenablement choisi. On voit alors facilement que $F(\lambda) = U_{f_0+\lambda\Delta f}$ est une fonction du paramètre numérique λ qui a une dérivée F'_λ égale à $\partial_{\Delta f}\,U_{f_0+\lambda\Delta f}$ pour λ assez petit. On a donc

$$F(\lambda) - F(0) = \lambda F'(\theta\lambda), \qquad \text{avec } 0 < \theta < 1$$

ou

$$U_{f_0+\lambda\Delta f} - U_{f_0} = \lambda\partial_{\Delta f}\,U_{f_0+\lambda\theta\Delta f}.$$

On aura une formule analogue en remplaçant U_f par $U_{f+\mu\Delta_1 f} - U_f$; d'où

$$(28)\quad U_{f_0+\lambda\Delta f+\mu\Delta_1 f} - U_{f_0+\lambda\Delta f} - U_{f_0+\mu\Delta_1 f} + U_{f_0} = \lambda\,[\partial_{\Delta f}\,U_{f_0+\mu\Delta_1 f+\lambda\theta\Delta f} - \partial_{\Delta f}\,U_{f_0+\lambda\theta\Delta f}]$$

pour

$$m(\mu\Delta_1 f + \lambda\Delta f) < \eta, \qquad m(\lambda\Delta f) < \eta,$$

ou

$$|\lambda|m\Delta f + |\mu|m\Delta_1 f < \eta.$$

Or on a en général, d'après la définition de la différentielle seconde,

$$\partial_{\Delta f}\,U_f - \partial_{\Delta f}\,U_{f_0} - \partial_{f-f_0}\,\partial_{\Delta f}\,U_{f_0} = w \cdot m(f - f_0),$$

où w tend vers zéro avec $m(f - f_0)$. En appliquant cette égalité pour $f = f_0 + \mu\Delta_1 f + \lambda\theta\Delta f$, et $f = f_0 + \lambda\theta\Delta f$, le second membre de l'égalité (28) devient

$$\lambda\,[\mu\partial_{\Delta_1 f}\,\partial_{\Delta f}\,U_{f_0} + w_1\,m(\mu\Delta_1 f + \lambda\theta\Delta f) - w_2\,m(\lambda\theta\Delta f)],$$

où w_1 et w_2 tendent vers zéro quand leurs facteurs respectifs tendent vers zéro. En particulier, pour $\mu = \lambda$, on a

$$\left|\frac{U_{f_0+\lambda(\Delta f+\Delta_1 f)} - U_{f_0+\lambda\Delta f} - U_{f_0+\lambda\Delta_1 f} + U_{f_0}}{\lambda^2} - \partial_{\Delta_1 f}\,\partial_{\Delta f}\,U_{f_0}\right|$$

$$= |\pm w_1\,m(\Delta_1 f + \theta\Delta f) \pm w_2\,m\theta\Delta f|$$

$$\leqq |w_1|(m\Delta_1 f + m\Delta f) + |w_2|m\Delta f,$$

d'où

$$\partial_{\Delta_1 f}\,\partial_{\Delta f}\,U_{f_0} = \lim_{\lambda=0}\frac{U_{f_0+\lambda(\Delta f+\Delta_1 f)} - U_{f_0+\lambda\Delta f} - U_{f_0+\lambda\Delta_1 f} + U_{f_0}}{\lambda^2}.$$

En permutant Δf et $\Delta_1 f$, le second membre ne change pas; on a donc

$$(29)\qquad \partial_{\Delta f}\,\partial_{\Delta_1 f}\,U_{f_0} = \partial_{\Delta_1 f}\,\partial_{\Delta f}\,U_{f_0}.$$

* Nous avons déjà donné une démonstration indirecte dans la communication citée plus haut.

16. D'après le théorème de F. Riesz, la différentielle première de U_f peut se mettre sous la forme

$$\partial_{\Delta f} U_f = \int_a^b \Delta f(s)\, d\alpha(s),$$

où $\alpha(s)$ est une fonction à variation bornée. On peut même supposer que α est une fonction partout régulière, nulle pour $s = b$. Alors ce sera une fonction bien déterminée pour chaque forme de f; on pourra la représenter par la notation $\alpha_f(s)$ et écrire

$$\partial_{\Delta f} U_f = \int_a^b \Delta f(s)\, d\alpha_f(s). \tag{30}$$

Nous savons que la différentielle seconde $\partial_{\Delta_1 f}\partial_{\Delta f} U_{f_0}$ est une fonctionnelle linéaire par rapport à Δ_1; il n'est pas évident qu'elle soit linéaire par rapport à Δf; *mais cela résulte de la formule* (29). Ainsi cette différentielle seconde est une fonctionnelle bilinéaire de Δf et de $\Delta_1 f$. D'après la théorie que nous avons développée, *on peut donc trouver une fonction $u(s,t)$ à variation bornée par rapport à l'ensemble de s et de t et par rapport à s et t séparément, telle qu'on puisse écrire*

$$\begin{aligned}(31)\quad \partial_{\Delta_1 f}\partial_{\Delta f} U_{f_0} &= \int_a^b \int_a^b \Delta f(s)\cdot \Delta_1 f(t)\cdot d_s\, d_t\, u(s,t)\\ &= \int_a^b \Delta f(s)\, d_s \int_a^b \Delta_1 f(t)\, d_t\, u(s,t)\\ &= \int_a^b \Delta_1 f(t)\, d_t \int_a^b \Delta f(s)\, d_s\, u(s,t).\end{aligned}$$

Mais puisque la différentielle seconde est symétrique en Δf, $\Delta_1 f$, les formules précédentes restent exactes si on remplace $u(s,t)$ par $u(t,s)$. Par suite, en prenant la demi-somme des deux systèmes de formules obtenues, on voit qu'on peut remplacer $u(s,t)$ par $[u(s,t)+u(t,s)]/2$; autrement dit, *on peut supposer que dans ces formules* (31), *$u(s,t)$ est une fonction symétrique de s et de t.*

Il serait intéressant et semble-t-il assez facile d'étudier la relation entre $\alpha_f(s)$ et $u(s,t)$.

ORIENTED CIRCLES IN SPACE*

BY

DAVID F. BARROW

INTRODUCTION

A wheel mounted on a spindle may be rotated in either of two directions. Thus rotation is a property which may be adjoined to a wheel in either of two ways. Analogously, *orientation* may be thought of as an arbitrary property which may be adjoined to any not null circle in two ways, thus giving rise to two *oriented circles*. In the mathematical development of this idea, two kinds of tangency present themselves, called proper and improper tangency. The study of oriented circles in two dimensions has been interestingly developed, especially by Laguerre, who discovered a one-to-one transformation of oriented circles that preserves one kind of tangency but not the other. In this paper I have considered oriented circles in space of three dimensions, and found a few theorems about invariants and linear systems. In one respect the results are disappointing, because all attempts to generalize the transformation of Laguerre were unsuccessful.

1. PENTASPHERICAL CONTINUUM†

All configurations considered in the following pages are supposed to lie in what may be called a *perfect pentaspherical continuum*, that is, a set of objects called *points* in one-to-one correspondence with all sets of five homogeneous coördinates, not all zero, $x_0 : x_1 : x_2 : x_3 : x_4$, which satisfy the relation

$$\sum_{i=0}^{4} x_i^2 \equiv (xx) = 0. \tag{1}$$

To have something definite in mind, let us observe that the points of finite cartesian space can be made into a perfect pentaspherical continuum by adjoining to them an infinite number of ideal points forming an ideal null sphere. Points that satisfy a linear relation

$$\sum_{i=0}^{4} y_i x_i \equiv (y\,x) = 0 \tag{2}$$

form a sphere. The five coefficients (y) are regarded as the coördinates of

* Presented to the Society, January 2, 1915.

† Darboux, *Sur une classe remarquable de courbes et de surfaces algébriques.*

the sphere. If (y) satisfies (1) the sphere is *null* and the point (y) is its *vertex*. In the extended cartesian space above alluded to, the isotropic planes appear as null spheres whose vertices are the ideal points. We use the same notation for point and sphere coördinates, but no confusion will arise from it.

The angle of two not null spheres (x) and (y) is found to be

$$\theta = \cos^{-1}\frac{(xy)}{\sqrt{(xx)}\sqrt{(yy)}}. \tag{3}$$

We shall study, in general, properties that are invariant under the group of conformal transformations of our space, which will be given by the orthogonal substitutions in the coördinates (x).

2. Plücker coördinates*

Two spheres, (x) and (y), will intersect in a circle. Consider the quantities

$$\lambda p_{ij} = (x_i y_j - x_j y_i) \tag{4}$$

where λ is a proportionality factor, not zero. If any other sphere of the pencil through this circle be used, e. g.,

$$lx_i + my_i \qquad (i = 0, 1, 2, 3, 4),$$

the effect will be merely to multiply every p_{ij} by the same factor. These quantities are called the plücker coördinates of the circle. They are connected by the following relations:

$$p_{ij} = -p_{ji}, \qquad p_{ii} = 0 \qquad (i, j = 0, 1, 2, 3, 4); \tag{5}$$

$$\begin{aligned}
\tfrac{1}{2}\Omega_0(pp) &\equiv p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23} = 0,\\
\tfrac{1}{2}\Omega_1(pp) &\equiv p_{02}p_{43} + p_{04}p_{32} + p_{03}p_{24} = 0,\\
\tfrac{1}{2}\Omega_2(pp) &\equiv p_{01}p_{34} + p_{03}p_{41} + p_{04}p_{13} = 0,\\
\tfrac{1}{2}\Omega_3(pp) &\equiv p_{01}p_{42} + p_{04}p_{21} + p_{02}p_{14} = 0,\\
\tfrac{1}{2}\Omega_4(pp) &\equiv p_{01}p_{23} + p_{02}p_{31} + p_{03}p_{12} = 0.
\end{aligned} \tag{6}$$

Conversely, every set of quantities p_{ij} not all zero, that satisfy (5) and (6), represent a circle. Of the twenty coördinates p_{ij} $(i \neq j)$, ten may be selected in several ways that are *essentially different*, that is, no two of them are connected by (5), e. g.,

$$p_{01}, p_{02}, p_{03}, p_{04}, p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34}.$$

The other ten are merely the negatives of these.

If a set of numbers p_{ij} and q_{ij} satisfy (5) alone, then the following identities

* Koenigs, *Contributions à la théorie des cercles dans l'espace*, Annales de la Faculté des Sciences de Toulouse, vol. 2 (1888).

will hold,

$$\sum_j p_{ij}\,\Omega_j(pp) \equiv 0 \qquad (i = 0, 1, 2, 3, 4), \tag{7}$$

$$\sum_j p_{ij}\,\Omega_j(qp) \equiv -\tfrac{1}{2}\sum_j q_{ij}\,\Omega_j(pp), \tag{8}$$

where by $\Omega_j(qp)$ is meant the polarized form of $\frac{1}{2}\Omega_j(pp)$. From (7) we see that only three independent conditions are imposed by (6). For suppose for example that $p_{34} \neq 0$. Then if

$$\Omega_0(pp) = \Omega_1(pp) = \Omega_2(pp) = 0,$$

it follows that

$$\Omega_3(pp) = \Omega_4(pp) = 0.$$

The condition that two circles be in involution, that is, that each be orthogonal to a sphere through the other, is

$$\sum_{ij} p_{ij}\,q_{ij} = 0. \tag{9}$$

The condition that two circles be cospherical is

$$\Omega_i(pq) = 0, \qquad (i = 0, 1, 2, 3, 4), \tag{10}$$

only two of which are independent conditions.

The condition that a circle be null* is

$$\sum p_{ij}^2 = 0. \tag{11}$$

The common orthogonal sphere of two non-cospherical circles is

$$\lambda x_i = \Omega_i(pq). \tag{12}$$

This is null and the two circles have a common point if

$$\sum_j [\Omega_j(pq)]^2 = 0. \tag{13}$$

Two circles are in bi-involution, that is every sphere through either is orthogonal to the other, if

$$\sum_n p_{in}\,q_{jn} = 0 \qquad (i, j = 0, 1, 2, 3, 4), \tag{14}$$

where only four of the equations are independent.

3. Orientation of circles

Let us introduce a redundant coördinate, p, defined by the relation

$$\sum p_{ij}^2 + p^2 = 0, \tag{15}$$

in which the summation is to be taken over only ten *essentially different* plücker coördinates, that is, no two of which are connected by (5). Two

* In the extended cartesian space to which we have alluded, in § 1, the isotropic lines appear as a special kind of null circle. They satisfy (11) but cannot be transformed into other null circles by any conformal transformation.

values of p, differing only in sign, go with every set of plücker coördinates except in case the circle is null when the redundant coördinate is zero on account of (11). Now let us say that in place of every not null circle, we have two *oriented circles that differ only in orientation.* In the real domain, it is customary to think of the two orientations of a circle as the two directions in which a moving point may describe it. In general, however, we may think of the orientation merely as an arbitrary property that may be adjoined to any not null circle in two ways. In the case of a null circle the two orientations are said to coincide. Thus we have a one-to-one correspondence between the oriented circles and the quantities p_{ij} and p, satisfying (5), (6), and (15), which are accordingly called the *coördinates* of the oriented circles.

But we can give a much more satisfactory meaning to the orientation of a circle. In the pencil of spheres through a not null circle are two null spheres whose vertices are called the *foci* of the circle. Let (x) and (y) denote the coördinates of the foci of a not null circle. The two null spheres through the circle have also these same coördinates (x) and (y). Let the *orientation* of a circle be a discrimination in favor of one of the foci, by saying that the circle whose *first focus* is (x) and *second focus* is (y) differs in orientation from the circle whose *first focus* is (y) and *second focus* is (x). In the case of proper null circles the foci coincide and the two orientations coincide as they ought. If the circle is an improper null circle, consisting of two coincident isotropic lines, every sphere in the pencil through it is null, and the orientation becomes illusory; but we shall define the two orientations as coincident also in this case.

We shall establish definite connections between this definition of orientation and the choice of the sign of the redundant coördinate by the following formula which gives the coördinates in terms of the first focus (x) and the second focus (y):

$$\rho p_{ij} = x_i y_j - x_j y_i, \qquad \rho p = (xy). \tag{16}$$

We must, of course, show that this value of p satisfies (15); but first observe that if the rôles of (x) and (y) are interchanged, the sign of every p_{ij} is reversed, but p is unaltered, which, by the homogeneity, amounts to changing the sign of p alone. Equation (15) may be multiplied by 2 and written in the form

$$\sum_{i,j=0}^{4} p_{ij}^2 + 2p^2 = 0,$$

where the summation is taken over all coördinates and not merely over ten essentially different ones. Substituting the values given by (16) we obtain

$$\sum_{i,j=0}^{4} [x_i^2 y_j^2 - 2x_i x_j y_i y_j + x_j^2 y_i^2] + 2(xy)^2$$
$$= (xx)(yy) - 2(xy)^2 + (xx)(yy) + 2(xy)^2 \equiv 0,$$

because $(xx) = (yy) = 0$, according to (1).

It would be perfectly logical to take $p = -(xy)$ instead of as in (16). It would only have associated the opposite orientation or choice of foci with a given choice of the sign of p. However we shall cling to the association established by formula (16).

If the circle is an improper null circle we may take the (x) and (y) as any two of the null spheres passing through it, and (16) holds true because $(xy) = 0$, and hence $p = 0$ as it ought. But in the case of proper null circles, formula (16) becomes illusory by giving zero for every coördinate, since $(x) \equiv (y)$. We can generalize (16) in a manner to cover this case and also prove useful later on. Let (x) be the null sphere whose vertex is the first focus of a circle, and (z) *any* other sphere through the circle, and let (y) be the second focus. Then

$$y_i = \lambda x_i + \mu z_i \qquad (i = 0, 1, 2, 3, 4).$$

Substituting for (y) in (16), and setting $\rho/\mu = \sigma$, we obtain

$$\begin{aligned} \sigma p_{ij} &= x_i z_j - x_j z_i, \\ \sigma p &= (xz), \end{aligned} \tag{17}$$

where (x) is the first focus and (z) is any sphere through the circle other than (x). This formula is equivalent to (16) if the circle is not null, and evidently holds in all cases. For null circles we have

$$(xz) = 0 \quad \text{but} \quad (x) \not\equiv (z).$$

Conversely, if we are given a set of coördinates, p_{ij}, p that satisfy (5), (6), and (15), we can calculate the first focus (x) by the formula

$$\lambda x_i = p \sum_j p_{ij}\, \xi_j - \sum_j \sum_n p_{in}\, p_{jn}\, \xi_j \qquad (i = 0, 1, 2, 3, 4), \tag{18}$$

where (ξ) is any sphere not orthogonal to the circle. In order to prove this formula, let us assume that (x') is the first focus of the circle in question and (z) any other sphere through the circle, and seek to prove that

$$x_i = \mu x_i' \qquad (i = 0, 1, 2, 3, 4).$$

Now formula (17) gives the coördinates in terms of (x') and (z), and these we substitute in (18), recollecting $(x'\,x') = 0$. The result is

$$\begin{aligned} \lambda x_i &= (x'\,z) \sum_j (x_i' z_j - x_j' z_i)\, \xi_j - \sum_j \sum_n (x_i' z_n - x_n' z_i)(x_j' z_n - x_n' z_j)\, \xi_j \\ &= (x'\,z)[x_i'(z\xi) - z_i(x'\,\xi)] - x_i'(zz)(x'\,\xi) + x_i'(x'\,z)(z\xi) \\ &\qquad + z_i(x'\,z)(x'\,\xi) = [2(x'\,z)(z\xi) - (zz)(x'\,\xi)]\, x_i'. \end{aligned}$$

Q. E. D.

To obtain the second focus of the circle, we need only change the sign of the redundant coördinate in formula (18).

An improper null circle is in bi-involution with itself, that is, each sphere through it is orthogonal to every sphere through it. The conditions for this are formed from (14) by making $(p) \equiv (q)$, giving

$$\sum_n p_{in} p_{jn} = 0 \qquad (i, j = 0, 1, 2, 3, 4),$$

and furthermore p zero. Hence formula (18) is indeterminate, giving zeros on the right, as we should expect since there are no determined foci for an improper null circle.

4. Angle of two oriented circles

Consider two not null circles (p) and (q), both lying on a not null sphere (z), and for the present let us think of them as not oriented. They meet at some angle, or rather a pair of supplementary angles which we wish to calculate. Let (u) and (v) be the coördinates of two spheres orthogonal to (z) and meeting it in the circles (p) and (q) respectively. Then (u) and (v) meet at the same angles as (p) and (q). Forming the plücker coördinates of the two circles from (u), (z) and (v), (z), as in formula (4), we have, since $(uz) = (vz) = 0$,

$$\frac{\sum p_{ij} q_{ij}}{\sqrt{\sum p_{ij}^2}\sqrt{\sum q_{ij}^2}} = \begin{vmatrix} (uv) & (uz) \\ (zv) & (zz) \end{vmatrix} \Big/ \sqrt{\begin{vmatrix} (uu) & (uz) \\ (zu) & (zz) \end{vmatrix}} \sqrt{\begin{vmatrix} (vv) & (vz) \\ (zv) & (zz) \end{vmatrix}} = \frac{(uv)}{\sqrt{(uu)}\sqrt{(vv)}},$$

where the summations are to be extended over only ten essentially different terms. By (3) this is the cosine of the angle made by (u) and (v), and so it is also the cosine of the angle made by (p) and (q), the choice of the sign of the denominator determining which of the two supplementary angles we have. Suppose we now orient the circles (p) and (q) by giving the redundant coördinates p and q, and define their angle θ to be

$$\theta = \cos^{-1} \frac{\sum p_{ij} q_{ij}}{-pq}.$$

What we have here done is to give a law by which one of the two angles is determined when the orientations are given; and we choose the negative sign in the denominator so that an oriented circle will make a null angle with itself, as is seen by making (q) identical with (p) and attending to (15). Hence

$$2 \sin^2 \theta/2 = 1 - \cos \theta = \frac{\sum p_{ij} q_{ij} + pq}{pq}.$$

The circles are tangent if the angle is zero or π. We shall define them as *properly tangent* when their angle is zero, and *improperly tangent* when it is π. For proper tangency then

$$\sum p_{ij} q_{ij} + pq = 0.$$

The expression

$$\frac{\sum p_{ij}\, q_{ij}}{\sqrt{\sum p_{ij}^2}\sqrt{\sum q_{ij}^2}}$$

formed for *any* two not null circles, is an absolute invariant under the conformal transformations of our space. It was first studied by Koenigs,* and again by Coolidge.†

We wish to examine a similar absolute invariant, namely

$$\vartheta(pq) \equiv \frac{\sum p_{ij}\, q_{ij} + pq}{pq},$$

the summation extending over only ten terms. This is defined uniquely for any two not null circles. Let us generalize the angle, θ, of any two not null oriented circles (p) and (q) by the definition

$$\tfrac{1}{2}\sin^2\theta/2 = \vartheta(pq). \tag{19}$$

As we saw, when the two circles lie upon a not null sphere, θ is a particular one of the two supplementary angles at which they intersect. In the general case the following construction gives an angle equal to θ.

Pass a sphere through (p) tangent to (q), and pass another sphere through (q) orthogonal to the first sphere. These two spheres meet in a circle, (r), tangent to (q) and cospherical with (p). Let (r) be oriented so as to be properly tangent to (q); then the angle made by (r) and (p) is equal to θ.

Now two spheres may be passed through (p) tangent to (q), and furthermore the rôles of (p) and (q) may be interchanged, so that this construction can be made in four ways. All four angles are proved equal to θ as follows: Calling ϕ the angle of (r) and (p), and recollecting that (r) and (p) are cospherical, we have $\frac{1}{2}\sin^2\phi/2 = \vartheta(pr)$. Hence we need only prove that $\vartheta(pr) = \vartheta(pq)$. Both $\vartheta(pq)$ and the angle above constructed are invariant under the conformal transformations, and so we may begin by moving the circles into a convenient position relative to the coördinate system. Accordingly, let the sphere through (p) tangent to (q) be $(0, 0, 0, 0, 1)$, and that through (q) orthogonal to it $(0, 0, 0, 1, 0)$. Then by means of (4) we see that all plücker coördinates of (p) are zero except those with the subscript 4, and all of (q) are zero except those with the subscript 3. Hence

$$\vartheta(pq) = \frac{p_{34}\, q_{34} + pq}{pq}.$$

The oriented circle (r), being the intersection of the two spheres given above,

* Loc. cit., p. 79.

† *A Study of the circle cross*, these Transactions, vol. 14 (1913), p. 149.

has but one plücker coördinate not zero, r_{34}. By (15),

$$r_{34}^2 + r^2 = 0.$$

Since (r) and (q) are properly tangent,

$$q_{34}\, r_{34} + qr = 0.$$

Multiplying by r and substituting $-r_{34}^2$ for r^2, we obtain

$$q_{34}\, rr_{34} - qr_{34}^2 = 0, \quad \text{or} \quad q_{34} = r_{34}/r \cdot q.$$

Hence

$$\vartheta(pq) = \frac{p_{34}\, q_{34} + pq}{pq} = \frac{p_{34}\, r_{34}/r \cdot q + pq}{pq} = \frac{p_{34}\, r_{34} + pr}{pr} = \vartheta(pr). \quad \text{Q. E. D.}$$

Now we fixed upon no special one of the four constructions. The same proof holds for all four. This establishes the uniqueness of the construction of the angle, and completes the proof of the theorem.

The construction for the angle is seen to yield, in the case of cospherical circles on a not null sphere, the angle at which they intersect, as it ought. In this case, a sphere through (p) tangent to (q) is merely the sphere on which (p) and (q) lie, and the constructed circle (r) coincides with (q). As to circles lying upon the same null sphere, which case has not been covered yet, recourse to formula (16) shows that two not null circles with the same first focus, or the same second focus, have a null angle; and if the first focus of one coincides with the second focus of the other, their angle is π. This is according to the definition by formula (19). But two circles, in fact, any two curves upon a null sphere considered in cartesian space meet at a null or a straight angle. For their angle is that made by their two tangent lines at their point of intersection. These two lines lie in an isotropic plane tangent to the null sphere, and so make a null or straight angle. This lends reasonableness to the definition by formula (19). Finally, if the circles are in bi-involution, the construction for their angle falls through; for all spheres through either are orthogonal to the other. Formula (19) gives their angle as a right angle, as we can prove by use of (14).

Our definition of proper and improper tangency was given only in case of circles on a not null sphere. We can now extend this definition:

If two not null circles are tangent in the sense of meeting in two coincident points, then they are *properly tangent* if their angle is null, and *improperly tangent* if their angle is π.

5. Some miscellaneous theorems

Formula (19) confronted by (14) yields

Theorem 1. *Two not null circles in involution make a right angle.*

Every linear combination of two oriented circles

$$r_{ij} = \lambda p_{ij} + \mu q_{ij}, \qquad r = \lambda p + \mu q,$$

will be the coördinates of an oriented circle, i. e. satisfy (5), (6), and (15), if and only if

$$\Omega_i(pq) = 0 \quad (i = 0, 1, 2, 3, 4), \qquad \sum p_{ij}\, q_{ij} + pq = 0.$$

Thus (p) and (q) must be cospherical and make a null angle if their angle be defined. If they lie upon a not null sphere, they are properly tangent and (r) sweeps out the properly tangent pencil determined by (p) and (q). But if they have the same first focus, or same second focus, they satisfy the conditions, and (r) sweeps out the pencil, not necessarily tangent, determined by (p) and (q) upon the null sphere which is the common focus. In fact if we take four oriented circles all having the same first focus (or same second focus), each two will satisfy the above conditions, and so the values (r), linear combinations of the four, will be the coördinates of oriented circles; and (r) will sweep out every oriented circle having that first (or second) focus.

An invariant of four oriented circles which should receive mention is

$$\frac{[\sum p_{ij}\, q_{ij} + pq]\,[\sum r_{ij}\, s_{ij} + rs]}{[\sum p_{ij}\, s_{ij} + ps]\,[\sum r_{ij}\, q_{ij} + rq]} = \frac{\sin^2 \tfrac{1}{2}\theta_{pq} \sin^2 \tfrac{1}{2}\theta_{rs}}{\sin^2 \tfrac{1}{2}\theta_{ps} \sin^2 \tfrac{1}{2}\theta_{rq}}.$$

If any of the circles is null the invariant still exists but cannot be expressed in terms of the angles which are not determined for null circles.

THEOREM 2. *The locus of the first foci of the oriented circles on a given not null sphere that cut a given oriented circle thereof at a fixed angle, is a sphere through the given circle.*

Let (z) be the given sphere and (α) the first focus of the given oriented circle (q), and (x) the first focus of an oriented circle (p) on (z) cutting (q) at the fixed angle $\theta = \cos^{-1} k$. Then

$$\cos\theta = \frac{\sum p_{ij}\, q_{ij}}{-pq} = \frac{(x\alpha)(zz) - (z\alpha)(xz)}{-(xz)(\alpha z)} = k,$$

or

$$(zz)(x\alpha) + (k-1)(z\alpha)(xz) = 0.$$

Hence (x) satisfies a linear relation and generates a sphere.

THEOREM 3. *Given six independent homogeneous equations in the coördinates p_{ij} and p, of degrees $n_1, n_2, n_3, n_4, n_5, n_6$ respectively; in general there will be just $10 n_1 n_2 n_3 n_4 n_5 n_6$ oriented circles whose coördinates satisfy the conditions.*

Now equations (6) constitute only three independent conditions. If we solve the first three equations of (6) together with the given equations and (5) and (15), we obtain $16 n_1 n_2 n_3 n_4 n_5 n_6$ solutions. All of these for which $p_{34} \neq 0$ will satisfy the last two equations of (6) as we saw by (7). But if any of these solutions have $p_{34} = 0$ they must be rejected for they will not in general satisfy the last two equations of (6), since these are independent of

the first three when $p_{34} = 0$. Solve

$$\Omega_0(pp) = \Omega_1(pp) = p_{34} = 0$$

together with (5), (15) and the six given equations, finding $8n_1 n_2 n_3 n_4 n_5 n_6$ solutions. By (7) these are seen to satisfy

$$p_{32}\,\Omega_2(pp) = 0, \qquad p_{42}\,\Omega_2(pp) = 0,$$

so they are divided into two categories: those that satisfy $\Omega_2(pp) = 0$ and those that satisfy $p_{32} = p_{42} = 0$. Finally solve

$$p_{32} = p_{42} = p_{34} = 0$$

with (6), (15), and the given equations, and get $2n_1 n_2 n_3 n_4 n_5 n_6$ solutions. These evidently make $\Omega_0(pp) = \Omega_1(pp) = 0; \Omega_2(pp) \neq 0$. Thus there are $6n_1 n_2 n_3 n_4 n_5 n_6$ solutions for which

$$\Omega_0(pp) = \Omega_1(pp) = \Omega_2(pp) = p_{34} = 0$$

and for these

$$\Omega_3(pp)\Omega_4(pp) \neq 0. \qquad \text{Q. E. D.}$$

Suppose we select p and any ten essentially different plücker coördinates, and regard them as the homogeneous coördinates of a point in ten-dimensional space. Then Theorem 3 shows that oriented circles correspond to a six-dimensional spread of tenth order lying in the ten-dimensional space. We shall denote this ten-dimensional space by S_{10}; and the six-dimensional spread of tenth order by S_6^{10}.

6. The linear hypercomplex

All oriented circles whose coördinates satisfy a single linear homogeneous relation shall be said to form a *linear hypercomplex*. The general linear equation contains twenty-one terms, but by aid of (5) twenty of these combine in pairs and yield

$$\sum a_{ij}\, p_{ij} + ap = 0, \tag{20}$$

where the summation extends over only ten essentially different terms. The hypercomplex is called *reduced* in case $a = 0$, and is called *special* if $\Omega_i(aa) = 0$ $(i = 0, 1, 2, 3, 4)$. If the hypercomplex is neither reduced nor special, and if furthermore $\sum [\Omega_i(aa)]^2 \neq 0$, then it is said to be *general.*

The linear hypercomplex corresponds in S_{10} to the points of S_6^{10} in a hyperplane. A hyperplane is determined by ten of its points, hence

Theorem 4. *Ten arbitrary oriented circles determine in general one linear hypercomplex of oriented circles whose coördinates are linearly dependent on those of the ten.*

In a special linear hypercomplex the coefficients a_{ij} satisfy the conditions

for being the coördinates of a non-oriented circle. If we orient this circle in both ways by adjoining the proper redundant coördinates $\pm\sqrt{-\sum a_{ij}^2}$, we obtain two circles that differ only in orientation, and are called the *central circles* of the special hypercomplex. If the central circles are not null, we may write the special hypercomplex in the form

$$\frac{\sum a_{ij}\,p_{ij} \pm p\sqrt{-\sum a_{ij}^2}}{\pm p\sqrt{-\sum a_{ij}^2}} = 1 - \frac{a}{\pm\sqrt{-\sum a_{ij}^2}}.$$

THEOREM 5. *The oriented circles of a special linear hypercomplex whose central circles are not null make fixed supplementary angles with two oriented circles that differ only in orientation.*

No two circles that differ only in orientation can belong to the same non-reduced linear hypercomplex. The not null oriented circles that belong to a reduced hypercomplex fall into pairs that differ only in orientation.

By aid of (17), (20) may be written

$$\sum a_{ij}\,(x_i\,z_j - x_j\,z_i) + a\,(xz) = 0,$$

summed over only ten terms. For convenience let us introduce ten new numbers a_{ji} defined by $a_{ji} = -a_{ij}$ and the above equation becomes

$$\sum_{i,\,j=0}^{4} a_{ij}\,x_i\,z_j + a\,(xz) = 0,$$

summed over all terms. Set

$$\xi_i = \textstyle\sum_j a_{ij}\,z_j \qquad (i = 0, 1, 2, 3, 4).$$

Then $(\xi z) = 0$ and $\sum_n \xi_n\,\Omega_n\,(aa) = 0$.

If (z) is held fixed, the focus (x) traces the sphere $\xi_i + az_i$ and by theorem 2, we have

THEOREM 6. *The oriented circles of a general linear hypercomplex upon an arbitrary not null sphere cut an oriented circle thereof at a certain angle which may differ for different spheres but is never a right angle.*

If the angle were ever a right angle we should be able to find two oriented circles of the hypercomplex differing only in orientation, which can only happen when the complex is reduced. If the hypercomplex is reduced, its circles that lie on an arbitrary sphere cut a circle thereof at right angles.

Suppose in the preceding work that (z) is null. Then (x) and (z) are the first and second foci of the circle (p). If we give (z) a fixed value (α), then (x) traces the sphere

$$u_i = \textstyle\sum_j a_{ij}\,\alpha_j + a\alpha_i.$$

If, on the other hand, we give (x) a fixed value (α), then (z) traces the

sphere

$$v_j = \sum_i a_{ij}\,\alpha_i + a\alpha_j = -\sum_i a_{ji}\,\alpha_i + a\alpha_j .$$

Now (u) and (v) are tangent at the point (α). Hence

THEOREM 7. *The oriented circles of a linear non-reduced hypercomplex upon a null sphere fall into two classes: if the vertex of the null sphere be the first focus the second focus traces a sphere; if it be the second focus, the first focus traces a sphere. These two spheres are tangent to each other at the vertex of the null sphere.*

*The central sphere** of the hypercomplex shall be that sphere whose coördinates are the five quantities $\Omega_i(a\alpha)$.

The reasoning leading to Theorem 6 fails if $\xi_i \equiv 0$, or $\xi_i \equiv \lambda z_i$. The former case occurs only if (z) is the central sphere. The latter case occurs if (z) is any one of four particular null spheres. In both cases the circles of the hypercomplex on (z) are all the null circles on (z). But in the case of the reduced hypercomplex every circle on the central sphere belongs to the hypercomplex.

The circles orthogonal to two spheres will pass through two points, namely the vertices of the circle in which these spheres meet. To investigate the circles of the hypercomplex that pass through two points we may examine those that are orthogonal to the two spheres $(0, 0, 0, 0, 1)$, and $(0, 0, 0, 1, 0)$ which need have no special relation to the hypercomplex. The conditions imposed on the circles are that every coördinate whose subscript contains either 3 or 4 shall be zero. These conditions carry with them the satisfaction of (6); so that only (15) and (20) remain, and we have

$$p_{01}^2 + p_{02}^2 + p_{12}^2 + p^2 = 0, \qquad a_{01}\,p_{01} + a_{02}\,p_{02} + a_{12}\,p_{12} + ap = 0.$$

These circles generate a surface. Now equation (13) is of the second order, and if solved with these yields four solutions for (p), showing that four of these circles meet an arbitrary circle (q). The surface in question meets a circle as often as it meets a line, hence

THEOREM 8. *The oriented circles of a general linear hypercomplex through two arbitrary points generate a two horned cyclide.*

The two horns or conical points are, of course, the two fixed points through which the circles pass.

Let θ and ϕ denote the angles that a variable oriented circle makes with two fixed not null oriented circles (α) and (β); and consider the oriented circles that satisfy

$$A\cos\theta + B\cos\phi + C = 0,$$

* G. Koenigs, loc. cit., page F. 12; also G. Castelnuovo, Atti della R. Istituto Veneto., ser. 7, vol. 2, part 1, page 861. This article is on line geometry in four dimensions. But this subject is equivalent to the study of non-oriented circles in three-way space; for a sphere may be regarded as a point of a four-dimensional space, and the pencil of spheres through a circle will correspond to a line in four dimensions.

where A, B, C, are constants. The oriented circles (p) that satisfy this generate a linear hypercomplex, for it may be written

$$A\frac{\sum\alpha_{ij}\,p_{ij}}{-\alpha p}+B\frac{\sum\beta_{ij}\,p_{ij}}{-\beta p}+C=0,$$

$$\sum[A\beta\alpha_{ij}+B\alpha\beta_{ij}]\,p_{ij}-C\alpha\beta p=0.$$

The central sphere of this hypercomplex is

$$z_i=2AB\alpha\beta\,\Omega_i(\alpha\beta) \qquad (i=0,1,2,3,4).$$

This sphere is orthogonal to the circles (α) and (β), for, by the identities (8) we obtain

$$\sum_n z_n\,\alpha_{in}=2AB\alpha\beta\sum_n\alpha_{in}\,\Omega_n(\alpha\beta)\equiv -AB\alpha\beta\sum_n\beta_{in}\,\Omega_n(\alpha\alpha)=0 \qquad (i=0,1,2,3,4);$$

and similarly for (β).

THEOREM 9. *If an oriented circle so move that the cosines of its angles with two fixed oriented circles satisfy a general linear relation, it traces a linear hypercomplex whose central sphere is the common orthogonal sphere of the two circles.*

Conversely, any linear hypercomplex can be described in this way. The problem is, given the hypercomplex (a), to find two suitable oriented circles (α) and (β). They must be orthogonal to the central sphere, by Theorem 9. First suppose that the central sphere of the complex is not null, and take for this sphere $(0,0,0,0,1)$. Then by (7) we have

$$a_{04}=a_{14}=a_{24}=a_{34}=0.$$

Also all coördinates of (α) and (β) with the subscript 4 are zero; and so all of equations (6) are satisfied automatically by them except the last one. Our problem reduces to finding (α) and (β) such that

$$\lambda\alpha_{ij}+\mu\beta_{ij}=a_{ij} \qquad (i,j=0,1,2,3),$$

subject to the further conditions

$$\Omega_4(\alpha\alpha)=\Omega_4(\beta\beta)=0.$$

Eliminating (β) we obtain

$$1/\mu^2\,\Omega_4(aa)-2(\lambda/\mu^2)\,\Omega_4(a\alpha)=0, \qquad \lambda=\Omega_4(aa)/2\Omega_4(a\alpha),$$

$$\Omega_4(a\alpha)\neq 0.$$

Hence

$$\beta_{ij}=\Omega_4(aa)\,\alpha_{ij}-2\Omega_4(a\alpha)\,a_{ij}.$$

Thus (α) may be chosen and (β) found to satisfy the demands.

Again, suppose that the central sphere of the given hypercomplex is null. We may take for its coördinates $(0,0,0,1,\sqrt{-1})$. Then by (7)

$$a_{34}=0, \qquad a_{i3}+\sqrt{-1}\,a_{14}=0 \qquad (i=0,1,2).$$

Similar relations will be imposed on the coördinates of (α) and (β); and in view of them, the first three equations of (6) are satisfied, and the last two reduce to the same condition. Our problem reduces to exactly what it was before, and can be solved as before. Again, if the hypercomplex is special, the coefficients are the plücker coördinates of some circle. In this case it is sufficient to take the (α) and (β) cospherical and such that the central circles of the hypercomplex belong to their (not tangent) pencil. Finally if the hypercomplex is $p = 0$, it consists of all the null circles, and this generation fails.

THEOREM 10. *Any linear hypercomplex except that of all null circles can be generated in an infinite number of ways by an oriented circle so moving that the cosines of its angles with two fixed oriented circles satisfy a general linear relation.*

Let us return to the general case where the central sphere is not null, and enquire whether the two circles can be in bi-involution. The conditions are

$$\beta_{01} = \rho\alpha_{23}, \quad \beta_{02} = \rho\alpha_{31}, \quad \beta_{03} = \rho\alpha_{12}, \quad \beta_{23} = \rho\alpha_{01}, \quad \beta_{31} = \rho\alpha_{02}, \quad \beta_{12} = \rho\alpha_{03},$$

which may be extracted from formula (14). In short, we seek a single circle (α) such that $\lambda\alpha_{ij} + \mu'\alpha_{kl} = a_{ij}$. Hence

$$\lambda\alpha_{kl} + \mu'\alpha_{ij} = a_{kl}, \qquad \alpha_{ij} = 1/(\lambda^2 - \mu'^2)(\lambda a_{ij} - \mu' a_{kl}),$$

and we must have

$$0 = \Omega_4(\alpha\alpha) = 1/(\lambda^2 - \mu'^2)^2 [\lambda^2 \Omega_4(aa) - 2\lambda\mu' \Sigma a_{ij}^2 + \mu'^2 \Omega_4(aa)].$$

Set

$$H = \frac{\sum a_{ij}^2}{\Omega_4(aa)}.$$

Then there are two values of $\lambda : \mu'$ and so two values for (α) namely

$$\alpha'_{ij} = [H + \sqrt{H^2 - 1}]\, a_{ij} - a_{kl}, \qquad \alpha''_{ij} = [H - \sqrt{H^2 - 1}]\, a_{ij} - a_{kl}.$$

These two circles form a *cross.** If (α) is chosen as one of them (β) must be the other. This circle cross may be used to define the hypercomplex as in Theorem 9, and is uniquely determined in general except, of course, for the orientation.

By a process similar to that used to deduce Theorem 9 we have

THEOREM 11. *If an oriented circle move so that the ratio of its invariants with two fixed oriented circles is constant, it describes a linear hypercomplex the central sphere of which is the common orthogonal sphere of the fixed circles.*

Can the general hypercomplex be generated in this way? As in the work leading up to Theorem 10, we first take the central sphere of the hypercomplex

* Coolidge first defined the circle cross, these Transactions, vol. 14 (1913), p. 149.

(a) as $(0, 0, 0, 0, 1)$, and then seek two oriented circles (α) and (β) such that

$$\lambda\alpha_{ij} + \mu\beta_{ij} = a_{ij} \quad (i, j = 0, 1, 2, 3), \qquad \lambda\alpha + \mu\beta = a,$$

where

$$\Omega_4(\alpha\alpha) = \Omega_4(\beta\beta) = 0, \qquad \sum\alpha_{ij}^2 + \alpha^2 = \sum\beta_{ij}^2 + \beta^2 = 0.$$

We can eliminate (β) in two ways this time and obtain

$$\Omega_4(\beta\beta) = \frac{1}{\mu^2}\Omega_4(aa) - 2\frac{\lambda}{\mu^2}\Omega_4(a\alpha) = 0,$$

$$\sum\beta_{ij} + \beta^2 = \frac{1}{\mu^2}[\sum a_{ij} + a^2] - \frac{2\lambda}{\mu}[\sum a_{ij}\alpha_{ij} + a\alpha] = 0.$$

Eliminating λ from these two equations gives

$$[\sum a_{ij}^2 + a^2]\,\Omega_4(a\alpha) - \Omega_4(aa)\,[\sum a_{ij}\alpha_{ij} + a\alpha] = 0,$$

assuming

$$\Omega_4(a\alpha) \neq 0, \qquad \sum a_{ij}\alpha_{ij} + a\alpha \neq 0.$$

Hence if we take (α) as a not null circle orthogonal to the central sphere and satisfying these conditions, another circle (β) can be found as desired. We find small interest in pushing the theorem to cases other than the general one.

THEOREM 12. *A general linear hypercomplex can be generated in an infinite number of ways by an oriented circle so moving that its invariants with two oriented circles orthogonal to the central sphere bear a constant ratio.*

7. THE LINEAR HYPERCONGRUENCE

The totality of oriented circles whose coördinates satisfy two independent linear conditions is called a *linear hypercongruence*, namely

$$\sum a_{ij}\,p_{ij} + ap = 0, \qquad \sum b_{ij}\,p_{ij} + bp = 0. \tag{21}$$

The hypercongruence corresponds to the intersection of S_6^{10} and two hyperplanes of S_{10}. Hence

THEOREM 13. *Nine arbitrary oriented circles determine in general one linear hypercongruence of oriented circles whose coördinates are linearly dependent on those of the nine.*

The oriented circles of a hypercongruence all belong to the *pencil* of linear hypercomplexes

$$l_{ij} = \lambda a_{ij} + \mu b_{ij}, \qquad l = \lambda a + \mu b.$$

THEOREM 14. *The oriented circles of a linear hypercongruence belong to one reduced hypercomplex.*

If, however, a and b are both zero, then every hypercomplex in the pencil is reduced.

The central sphere* of (l) is

$$\Omega_i(ll) = \lambda^2\,\Omega_i(aa) + 2\lambda\mu\Omega_i(ab) + \mu^2\,\Omega_i(bb).$$

THEOREM 15. *If two hypercomplexes have the same central sphere this is also the central sphere of every hypercomplex of their pencil.*

The proof of this theorem is immediate if the common central sphere be taken as $(0, 0, 0, 0, 1)$.

THEOREM 16. *If two linear hypercomplexes are special, making fixed angles with two circles, their pencil has one common central sphere orthogonal to the two circles, unless these two circles be cospherical when all hypercomplexes of the pencil are special, and consist of oriented circles making fixed angles with the circles of the pencil determined by the two.*

To prove this theorem observe that if $\Omega_i(aa) = \Omega_i(bb) = 0$ $(i = 0, 1, 2, 3, 4)$, then $\Omega_i(ll) = 2\lambda\mu\Omega_i(ab)$. And if furthermore $\Omega_i(ab) = 0$, then $\Omega_i(ll) = 0$.

THEOREM 17. *If one hypercomplex be special and another not, then the central spheres of their pencil generate a coaxial system.*

THEOREM 18. *If two linear hypercomplexes have different central spheres, then their central spheres generate a one-parameter family whereof two are orthogonal to an arbitrary sphere, called a conic series.*

The envelope of the series is the cyclide

$$\sum\Omega_i(aa)\,x_i \sum\Omega_i(bb)\,x_i - [\sum\Omega_i(ab)\,x_i]^2 = 0.$$

If $\Omega_i(ab) = 0$, $(i = 0, 1, 2, 3, 4)$, then $\rho\Omega_i(aa) + \sigma\Omega_i(bb) = 0$. For by means of the identities (7) and (8)

$$\sum_n a_{in}\,\Omega_n(aa) = 0, \qquad \sum_n a_{in}\,\Omega_n(bb) = 0$$

$$\sum_n b_{in}\,\Omega_n(aa) = 0, \qquad \sum_n b_{in}\,\Omega_n(bb) = 0,$$

and $\Omega_i(aa)$ and $\Omega_i(bb)$ are determined by the same set of linear equations. Hence the pencil of hypercomplexes have the same central sphere, or else all of them are special.

By Theorem 6 we observe that the oriented circles of a linear hypercongruence upon an arbitrary sphere cut two oriented circles thereof at fixed angles. Hence they are tangent to two circles in the pencil of these two.

THEOREM 19. *The oriented circles of a linear hypercongruence upon an arbitrary not null sphere are properly tangent to two oriented circles thereof.*

By aid of Theorem 7, we obtain

THEOREM 20. *The oriented circles of a linear hypercongruence upon an arbitrary null sphere fall into two classes; if the vertex of the null sphere be the*

*The central spheres of the hypercongruenee are studied by G. Castelnuovo, loc. cit., p. 863 ff.

first focus, the second focus traces a circle; if it be the second focus, the first focus traces a circle. These two circles are tangent at the vertex of the null sphere.

Let us examine for exceptions to Theorem 19. By means of Theorem 14, we may regard the general hypercongruence as determined by one reduced and one non-reduced hypercomplex. But as we remarked, the reduced hypercomplex imposes no condition upon the circles of its central sphere. Hence

THEOREM 21. *The oriented circles of a general hypercongruence upon the central sphere of the reduced hypercomplex of the corresponding pencil are those cutting an oriented circle thereof at a fixed angle.*

We observe further that the oriented circles of a non-reduced hypercomplex upon its central sphere are all null.

THEOREM 22. *The oriented circles of the general hypercongruence upon the central sphere of a non-reduced hypercomplex of the pencil are all null and their centers trace a circle upon the sphere.*

This theorem may be regarded as the limiting case of Theorem 19 where the oriented circles are all tangent to two oriented circles that differ only in orientation. Thus Theorem 21 mentions the only not null sphere that has a two parameter family of the oriented circles of the hypercongruence on it. For in every other case the reduced hypercomplex requires the circles to cut a circle at a right angle, and the non-reduced hypercomplex requires them to cut an oriented circle at some other angle, and these two conditions must always be independent.

The locus of the centers of the null circles of the hypercongruence that lie upon the series of central spheres is easily found. In the proof preceding Theorem 6, we found that the first foci of the oriented circles of the hypercomplex (a), which lie upon an arbitrary sphere (z), trace a certain sphere (ξ) where

$$\xi_i = \sum_j a_{ij} z_j .$$

Now in our case we take for (z) the central sphere of the general hypercomplex of the pencil, (l). Then the null circles of this sphere that belong to the hypercomplex (a) must have their centers lying on the sphere

$$\xi_i = \sum_j a_{ij} \Omega_j (ll) = \sum_j [\lambda^2 \Omega_j (aa) + 2\lambda\mu\Omega_j (ab) + \mu^2 \Omega_j (bb)] a_{ij} .$$

By using identities (7) and (8) we obtain

$$\xi_i = \mu \sum_j [\mu a_{ij} \Omega_j (bb) - \lambda b_{ij} \Omega_j (aa)] .$$

What we wish to do is to find the locus of the intersection of (ξ) and the central sphere of (l) as $\lambda : \mu$ varies. We need only eliminate $\lambda : \mu$ between

$$(\xi x) = 0, \qquad \sum_i \Omega_i (ll) x_i = 0,$$

and we obtain the equation

$$[\sum_{ij} a_{ij}\,\Omega_j(bb)\,x_i]^2 \sum_j \Omega_i(aa)\,x_i + 2[\sum_{ij} a_{ij}\,\Omega_j(bb)\,x_i][\sum_{ij} b_{ij}\,\Omega_j(aa)\,x_i]\sum_i \Omega_i(ab)\,x_i + [\sum_{ij} b_{ij}\,\Omega_j(aa)\,x_i]^2 \sum_i \Omega_i(bb)\,x_i = 0.$$

Consider the circles of the general linear hypercongruence orthogonal to $(0,0,0,0,1)$ and $(0,0,0,1,0)$, which spheres need bear no special relation to the hypercongruence. All coördinates with the subscript 3 or 4 are zero. Conditions (6) are all satisfied because of this. We have then equations (5), (15), and (21) to determine the remaining coördinates, which have two solutions. Hence

THEOREM 23. *Two circles of the general linear hypercongruence pass through an arbitrary pair of points.*

8. THE LINEAR COMPLEX

Oriented circles whose coördinates satisfy three independent linear relations generate a *linear complex*,

$$(22)\quad \sum a_{ij}\,p_{ij} + ap = 0, \qquad \sum b_{ij}\,p_{ij} + bp = 0, \qquad \sum c_{ij}\,p_{ij} + cp = 0.$$

These oriented circles correspond, of course, to the points of S_6^{10} that lie in a linear spread of seven dimensions.

THEOREM 24. *Eight arbitrary oriented circles determine in general one linear complex of oriented circles whose coördinates depend linearly upon those of the eight.*

The oriented circles of a linear complex are common to a *net* of hypercomplexes

$$l_{ij} = \lambda a_{ij} + \mu b_{ij} + \nu c_{ij}, \qquad l = \lambda a + \mu b + \nu c.$$

THEOREM 25. *The oriented circles of a complex belong to a pencil of reduced hypercomplexes.*

If a, b, and c are all zero, then all hypercomplexes of the net are reduced.

The central sphere* of (l) is

$$\Omega_i(ll) = \lambda^2\,\Omega_i(aa) + \mu^2\,\Omega_i(bb) + \nu^2\,\Omega_i(cc) + 2\lambda\mu\Omega_i(ab) + 2\lambda\nu\Omega_i(ac) + 2\mu\nu\Omega_i(bc).$$

THEOREM 26. *The central spheres of a linear set of hypercomplexes generate in general a quartic congruence of spheres.*

The envelope of this congruence is

$$\begin{vmatrix} \sum\Omega_i(aa)\,x_i & \sum\Omega_i(ab)\,x_i & \sum\Omega_i(ac)\,x_i \\ \sum\Omega_i(ba)\,x_i & \sum\Omega_i(bb)\,x_i & \sum\Omega_i(bc)\,x_i \\ \sum\Omega_i(ca)\,x_i & \sum\Omega_i(cb)\,x_i & \sum\Omega_i(cc)\,x_i \end{vmatrix} = 0.$$

THEOREM 27. *If three hypercomplexes have a common central sphere, then this is the central sphere for every hypercomplex of their net.*

* Central spheres are discussed by Castelnuovo, loc. cit., p. 868 ff.

The proof is immediate on taking for the common central sphere $(0, 0, 0, 0, 1)$.

By Theorem 25 we may regard the complex as determined by two reduced and one not reduced linear hypercomplex. Thus the oriented circles thereof that lie upon an arbitrary sphere must cut two circles thereof at right angles, and make a certain other angle with another oriented circle. Two circles do this, hence

THEOREM 28. *Two oriented circles of a complex lie upon an arbitrary sphere.*

If, however, we take the central sphere of one of the reduced hypercomplexes, its circles are subject to only the two conditions imposed by the other two hypercomplexes, and as we proved in Theorem 19, we obtain

THEOREM 29. *The oriented circles of the complex upon the central sphere of one of the reduced hypercomplexes are those properly tangent to two oriented circles.*

THEOREM 30. *Every null circle of the complex lies on three central spheres of the net that are mutually tangent, and when three central spheres are mutually tangent at a point, their common null circle belongs to the complex.*

The proof of this theorem comes from the fact that the null circles of the general complex are the same as the null circles of the complex reduced by setting a, b, and c all zero, in which every circle of the complex lies on three central spheres.*

Consider the circles of the general complex orthogonal to $(0, 0, 0, 0, 1)$, which sphere need bear no special relation to the complex. All coördinates with subscript 4 are zero. The remaining coördinates are subject to conditions (21), (15), (5), and the last equation of (6). Let us enquire how many of them meet an arbitrary circle (q). The condition as given by (13) is quadratic. Hence we find eight solutions.

THEOREM 31. *The surface swept out by the oriented circles of the complex orthogonal to an arbitrary sphere is of the eighth order, having in euclidean space the circle at infinity as a quadruple curve.*

9. THE LINEAR CONGRUENCE

Oriented circles whose coördinates satisfy four independent linear equations form a *linear congruence*

$$\begin{aligned} &\sum a_{ij}\, p_{ij} + ap = 0, \qquad \sum b_{ij}\, p_{ij} + bp = 0, \\ &\sum c_{ij}\, p_{ij} + cp = 0, \qquad \sum d_{ij}\, p_{ij} + dp = 0. \end{aligned} \tag{23}$$

These oriented circles correspond to the points of S_6^{10} lying in a linear subspace of six dimensions, hence

THEOREM 32. *Seven arbitrary oriented circles determine in general one congruence of oriented circles whose coördinates are linearly dependent upon those of the seven.*

* For proof of this see G. Castelnuovo, loc. cit., p. 870.

The oriented circles of the congruence are common to the three-parameter family of hypercomplexes linearly dependent upon four

$$l_{ij} = \lambda a_{ij} + \mu b_{ij} + \nu c_{ij} + \rho d_{ij}, \qquad l = \lambda a + \mu b + \nu c + \rho d.$$

By a method used to prove Theorem 3, we find that five sets of the quantities l_{ij} satisfy equations (6). These five sets of quantities are consequently the plücker coördinates of five non-oriented circles. This is the *pentacycle* of Stephanos.* Each of these circles may be given an orientation in two ways, whence result the theorems

THEOREM 33. *The general linear congruence consists in thirty-two ways of oriented circles making a (different) fixed angle with each circle of a pentacycle.*

THEOREM 34. *Oriented circles making fixed angles with four arbitrary oriented circles make a fixed angle with a fifth which completes the pentacycle. If each of the first four angles is a right angle, the fifth is also a right angle.*

We observe that a two-parameter net of the hypercomplexes of (l) are reduced.

THEOREM 35. *The oriented circles of a congruence are common to a net of reduced hypercomplexes.*

Now no circle of the congruence lies on a general sphere, but the oriented circles upon the central sphere of one of the reduced hypercomplexes are, as we have seen, subject to only three conditions and two circles of the congruence lie upon each.

THEOREM 36. *Two oriented circles of the congruence lie upon the central sphere of each of the reduced hypercomplexes to which the congruence belongs.*

These are all the oriented circles of the congruence. For our congruence belongs to a reduced linear complex determined by the net of reduced hypercomplexes mentioned in Theorem 35; and Castelnuovo has shown (loc. cit., p. 870) that the circles of such a complex all lie upon the net of central spheres.

The sphere $(0, 0, 0, 0, 1)$ will bear no special relation to the general linear congruence, and we see that four oriented circles of the congruence are orthogonal to it.

THEOREM 37. *Four oriented circles of a linear congruence are orthogonal to an arbitrary sphere.*

To find the locus of the first foci of the oriented circles of the congruence, we must use the formula for determining the plücker coördinates of a circle by means of three spheres, (r), (s), and (t), orthogonal to it, namely

$$\alpha_{ij} = \begin{vmatrix} r_k & r_l & r_m \\ s_k & s_l & s_m \\ t_k & t_l & t_m \end{vmatrix}.$$

Now by Theorem 35, we may use in place of the hypercomplex (d) of

*C. Stephanos, Comptes Rendus, vol. 93.

formula (23) a reduced hypercomplex to determine our congruence. And by Theorem 10 this reduced hypercomplex may be written

$$\sum \alpha_{ij}\, p_{ij} + \sum \beta_{ij}\, p_{ij} = 0,$$

where (α) and (β) are the plücker coördinates of two circles. Let these two circles be determined by the two triads of spheres, (r), (s), and (t); and (r'), (s'), and (t') orthogonal to them. Furthermore let the variable circle (p) be given in terms of its first and second foci, (x) and (y). The above equation then becomes

$$\begin{vmatrix} x_0 & x_1 & x_2 & x_3 & x_4 \\ y_0 & \cdot & \cdot & \cdot & y_4 \\ r_0 & \cdot & \cdot & \cdot & r_4 \\ s_0 & \cdot & \cdot & \cdot & s_4 \\ t_0 & \cdot & \cdot & \cdot & t_4 \end{vmatrix} + \begin{vmatrix} x_0 & \cdot & \cdot & \cdot & x_4 \\ y_0 & \cdot & \cdot & \cdot & y_4 \\ r_0' & \cdot & \cdot & \cdot & r_4' \\ s_0' & \cdot & \cdot & \cdot & s_4' \\ t_0' & \cdot & \cdot & \cdot & t_4' \end{vmatrix} = 0.$$

Denote the first determinant by Δ and the second by Δ'. Besides this equation we have the first three equations of (23), which in terms of (x) and (y) are

$$\sum_{i,j=0}^{4} a_{ij}\, x_i\, y_j + a\,(xy) = 0,$$

$$\sum_{i,j=0}^{4} b_{ij}\, x_i\, y_j + b\,(xy) = 0,$$

$$\sum_{i,j=0}^{4} c_{ij}\, x_i\, y_j + c\,(xy) = 0,$$

where the summation is to extend over all terms. Finally, because (x) and (y) are null spheres, we have

$$(xx) = (yy) = 0.$$

We wish to eliminate the second focus (y) and obtain the equation of the surface in (x). The first step is the pseudo-elimination of (y) by equating to zero the following determinant:

$$\begin{vmatrix} \dfrac{\partial \Delta}{\partial y_0} + \dfrac{\partial \Delta'}{\partial y_0} & \dfrac{\partial \Delta}{\partial y_1} + \dfrac{\partial \Delta'}{\partial y_1} & \cdots & \dfrac{\partial \Delta}{\partial y_4} + \dfrac{\partial \Delta'}{\partial y_4} \\ y_0 & y_1 & \cdots & y_4 \\ \sum_i a_{i0} x_i + a x_0 & \sum_i a_{i1} x_i + a x_1 & \cdots & \sum_i a_{i4} x_i + a x_4 \\ \sum_i b_{i0} x_i + b x_0 & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & & \sum_i b_{i4} x_i + b x_4 \\ \sum_i c_{i0} x_i + c x_0 & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & & \sum_i c_{i4} x_i + c x_4 \end{vmatrix} = 0.$$

This equation is true, but not yet free from (y). Let this determinant be expressed as the sum of two determinants, the first row of one having the elements $\partial\Delta/\partial y_i$, and the first row of the other $\partial\Delta'/\partial y_i$; the last four rows being as they stand above. Each of these determinants may be regarded as the

product of two matrices; for the elements of the first rows are themselves determinants taken from a certain matrix of four rows and five columns. Hence

$$\begin{vmatrix} (xy) & (ry) & (sy) & (ty) \\ 0 & \sum_{ij} a_{ij} x_i r_j + a(xr) & \cdots & \sum_{ij} a_{ij} x_i t_j + a(xt) \\ 0 & \sum_{ij} b_{ij} x_i r_j + b(xr) & \cdots & \sum_{ij} b_{ij} x_i t_j + b(xt) \\ 0 & \sum_{ij} c_{ij} x_i r_j + c(xr) & \cdots & \sum_{ij} c_{ij} x_i t_j + c(xt) \end{vmatrix}$$

$$+ \begin{vmatrix} (xy) & (r'y) & (s'y) & (t'y) \\ 0 & \sum_{ij} a_{ij} x_i t'_j + a(xr') & \cdots & \\ 0 & \cdots & \cdots & \\ 0 & \cdots & & \sum_{ij} c_{ij} x_i t'_j + c(xt') \end{vmatrix} = 0.$$

Factoring out (xy) we have the equation. Hence

THEOREM 38. *The locus of the first foci of the oriented circles of a linear congruence is a surface of the sixth order having in euclidean space the circle at infinity as a triple curve. The second foci trace another such locus, and the two surfaces fall together in case the hypercomplexes determining the congruence are all reduced.*

10. THE LINEAR SERIES

Oriented circles whose coördinates satisfy five independent linear relations form a *linear series,*

$$\begin{gathered} \sum a_{ij} p_{ij} + ap = 0, \qquad \sum b_{ij} p_{ij} + bp = 0, \qquad \sum c_{ij} p_{ij} + cp = 0, \\ \sum d_{ij} p_{ij} + dp = 0, \qquad \sum e_{ij} p_{ij} + ep = 0. \end{gathered} \tag{24}$$

THEOREM 39. *Six arbitrary oriented circles determine in general one linear series of oriented circles whose coördinates are linearly dependent on those of the six.*

The oriented circles of a linear series are common to a four-parameter family of hypercomplexes linearly dependent on five,

$$\begin{gathered} l_{ij} = \lambda a_{ij} + \mu b_{ij} + \nu c_{ij} + \rho d_{ij} + \sigma e_{ij}, \\ l = \lambda a + \mu b + \nu c + \rho d + \sigma e. \end{gathered}$$

If the given five hypercomplexes are reduced, every one of this system will be reduced, and the series will naturally be called *reduced.*

THEOREM 40. *The oriented circles of a non-reduced series belong to a three-parameter system of reduced hypercomplexes linearly dependent on four.*

By methods like those used to prove Theorem 3 we find that five members of this three parameter family are special hypercomplexes. Their central circles form a pentacycle, hence

THEOREM 41. *The oriented circles of a series make a right angle with each circle of a pentacycle, besides satisfying another independent condition.*

This pentacycle is in general unique.

There are an infinite number of special hypercomplexes in the four-parameter system (l). The plücker coördinates (but not the redundant coördinates) of the central circles of these are linearly dependent upon the five sets of coefficients in (24), and so they satisfy five reduced linear equations and generate a reduced series. By aid of Theorem 5, we obtain

THEOREM 42. *The oriented circles of a general linear series make fixed angles with the circles of a reduced series. This series contains every pentacycle of which it contains four circles.*

Five oriented circles determine in general one and only one reduced series.

The hypercomplexes (l), being linearly dependent upon five, will represent the points of a flat four-way spread in ten dimensions. This meets S_6^{10} in ten points by Theorem 3. In general, the ten points in which an arbitrary four-way spread meets S_6^{10} correspond to ten oriented circles which shall be called a *dekacycle*. The general dekacycle is determined by any five of its oriented circles, because five points determine a four-way spread; and the other five oriented circles are linearly dependent upon the chosen five. A special case of the dekacycle is obtained from the pentacycle by giving to each of its circles both orientations. The five sets of coefficients in (24) represent five points in S_{10}, and the four-way spread linearly dependent on them meets S_6^{10} in a dekacycle, hence

THEOREM 43. *The oriented circles of a general series are those that make a null angle with each oriented circle of a dekacycle.*

By Theorem 40 we may choose as reduced four of the five equations that determine a general series. In other words, (24) is not less general if we set

$$b = c = d = e = 0.$$

Having done this, set

$$a = a' - a''$$

and (24) may be written

$$\sum a_{ij}\, p_{ij} + a'\, p = a''\, p, \qquad \sum b_{ij}\, p_{ij} = 0, \qquad \cdots, \qquad \sum e_{ij}\, p_{ij} = 0.$$

Now there is a dekacycle dependent on a_{ij}, a'; b_{ij}; c_{ij}; d_{ij}; e_{ij}, which we may indicate thus

$$\alpha_{ij}^{(k)} = \lambda^{(k)}\, a_{ij} + \mu^{(k)}\, b_{ij} + \cdots + \sigma^{(k)}\, e_{ij}, \qquad \alpha^{(k)} = \lambda^{(k)}\, a'.$$

Hence

$$\sum \alpha_{ij}^{(k)}\, p_{ij} + \alpha^{(k)}\, p = \lambda^{(k)}\, a''\, p.$$

Dividing this equation by $\alpha^{(k)}\, p$ we obtain

$$\theta\,(\alpha^{(k)}\, p) = a''/a' \qquad (k = 1, 2, 3, \cdots, 10).$$

Now evidently a''/a' can be made to have any value we choose. Hence

THEOREM 44. *A dekacycle can be found each of whose circles makes the*

same arbitrary angle with the oriented circles of an arbitrary series. The infinite number of dekacycles thus determined by a given series generate a reduced series.

If the arbitrary angle is chosen as $\pi/2$, the dekacycle reduces to the pentacycle of Theorem 41, each of its circles having both orientations.

THEOREM 45. *A linear series contains every dekacycle of which it contains five oriented circles.* For a five-way spread contains every four-way spread of which it contains five points.

THEOREM 46. *Every dekacycle belongs in general to one and only one reduced series.*

An exception occurs if the dekacycle falls into two pentacycles that differ only in orientation, in which case an infinite number of reduced series contain the dekacycle.

The condition that two circles intersect as given by (13) is quadratic in the coördinates of either. Hence, by Theorem 3, twenty circles of the linear series will meet an arbitrary circle.

THEOREM 47. *The oriented circles of a linear series sweep out a surface of the twentieth order which has, in euclidean space, the circle at infinity as a tenfold curve.*

If the series be reduced, the surface doubles upon itself and becomes one of the tenth order, for, while twenty circles still meet an arbitrary circle, yet these are evidently divided into ten pairs that differ only in orientation.

The condition that an oriented circle be null is linear, $p = 0$. Hence by Theorem 3

THEOREM 48. *There are ten null circles in a linear series.*

Let (ξ) and (α) be two arbitrary spheres, and consider the equation

$$p\sum_{ij} p_{ij}\,\alpha_i\,\xi_j - \sum_{ijn} p_{in}\,p_{jn}\,\alpha_i\,\xi_j = 0.$$

On consulting (18) we observe that this equation is satisfied not only by all oriented circles which have the first focus orthogonal to (α) but also by all those which have the second focus orthogonal to (ξ). Moreover this equation is quadratic in (p), and so by Theorem 3 there are twenty oriented circles of a general linear series that satisfy it. By symmetry ten of these lie with the first focus orthogonal to (α), and the other ten with the second foci orthogonal to (ξ). Hence

THEOREM 49. *The first foci of the oriented circles of a linear series trace a curve of the tenth order, and so do the second foci. These two curves meet in ten points, the centers of the null circles of the series. If the series is reduced the two curves coincide.*

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A NEW ISOSCELES-TRIANGLE SOLUTION OF THE THREE BODY PROBLEM*

BY

DANIEL BUCHANAN

1. Introduction

The known isosceles-triangle solutions of the problem of three bodies are the periodic solutions in which two of the bodies have equal finite masses, while the third body moves in a straight line and remains equidistant from the other bodies. In my Chicago dissertation† three cases of the isosceles-triangle solutions were discussed. In the first case the finite bodies move in a circle and the third body is infinitesimal. The solution of this case was first obtained by Pavanini‡ by means of elliptic integrals, and was later developed independently by MacMillan.§ The solution which was obtained in my dissertation is analogous to that obtained by MacMillan. In the second case, not treated by Pavanini and MacMillan, the finite bodies move in ellipses and the third body is infinitesimal. In the third case, all three bodies are finite. The period of the motion in case III is the same as that in case I, and the solution reduces to the solution of the first case when the third body becomes infinitesimal.

The present paper is an extension of the second case of my dissertation to that in which all three masses are finite. Periodic orbits are here shown to exist, and a practical method is given of computing them. These solutions have the same period as those of case II of my dissertation, and reduce to them when the third body becomes infinitesimal.

2. The differential equations of motion

Let m_1 and m_2 represent the two finite bodies of mass m, and let the third body be denoted by μ. The origin of coördinates will be taken at the center of mass of the system. The plane perpendicular to the initial motion of μ

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† Moulton's *Periodic Orbits*, Chap. X.

‡ Annali di Matematica, ser. 3, vol. 13 (1907), pp. 179–202.

§ Astronomical Journal, Nos. 625–626 (1911).

will be taken as the $\xi\eta$-plane. Then let the coördinates of m_1, m_2, and μ be denoted by ξ_1, η_1, ζ_1; ξ_2, η_2, ζ_2; and ξ, η, ζ respectively.

When $\mu = 0$ the mean angular motion of the finite bodies is*

$$n = \frac{k\sqrt{2m}}{a^{3/2}}, \tag{1}$$

where k is the gravitational constant and a is the mean distance from m_1 to m_2. Now let the unit of mass be so chosen that $m = 1/2$, and let the linear and time units be so determined that k and a are each unity. With the units thus chosen $n = 1$.

We shall suppose that μ is started from the origin of coördinates at the initial time t_0 when m_1 and m_2 are at apsides of their orbits, which can be assumed, without loss of geometric generality, to lie on the ξ-axis. When expressed analytically, these initial conditions are

$$\xi = \eta = \zeta = \xi' = \eta' = \eta_1 = \eta_2 = \zeta_1 = \zeta_2 = 0 \quad \text{at} \quad t = t_0,$$

$$\xi_1(t_0) = -\xi_2(t_0) = \frac{a}{2}(1-e), \qquad \xi_1'(t_0) = -\xi_2'(t_0), \qquad \eta_1'(t_0) = -\eta_2'(t_0),$$

where e is the eccentricity of the ellipses in which m_1 and m_2 move when $\mu = 0$. Under these initial conditions the orbits described by m_1 and m_2 will be symmetrical; that is,

$$\xi_1 \equiv -\xi_2, \qquad \eta_1 \equiv -\eta_2, \qquad \zeta_1 \equiv \zeta_2. \tag{2}$$

On making use of (2) in the center of gravity equations, which are

$$m_1\xi_1 + m_2\xi_2 + \mu\xi = 0, \qquad m_1\eta_1 + m_2\eta_2 + \mu\eta = 0,$$

$$m_1\zeta_1 + m_2\zeta_2 + \mu\zeta = 0,$$

we obtain

$$\xi \equiv \eta \equiv 0, \qquad \zeta_1 + \mu\zeta = 0. \tag{3}$$

Hence the third body always remains on the ζ-axis.

The differential equations of motion when simplified by (3) become

$$\begin{aligned} \xi_1'' &= -\frac{\xi_1}{8r^3} - \frac{\mu\xi_1}{[r^2 + (1+\mu)^2\zeta^2]^{3/2}}, \\ \eta_1'' &= -\frac{\eta_1}{8r^3} - \frac{\mu\eta_1}{[r^2 + (1+\mu)^2\zeta^2]^{3/2}}, \\ \zeta'' &= \frac{-2m(1+\mu)\zeta}{[r^2 + (1+\mu)^2\zeta^2]^{3/2}}, \end{aligned} \tag{4}$$

where $r^2 = \xi_1^2 + \eta_1^2$. When these equations have been integrated, the solu-

* Moulton's *Celestial Mechanics*, p. 142.

tions for ξ_2, η_2, ζ_1, and ζ_2 can be obtained from (2) and (3). The numerical value of $m = 1/2$ has been substituted in the first two equations of (4) but not in the last equation. The reason for this will appear in the next section.

On transforming to polar coördinates by the substitutions

$$\xi_1 = r\cos v, \qquad \eta_1 = r\sin v, \qquad \zeta = z,$$

the differential equations become

$$\begin{aligned} r'' - r(v')^2 + \frac{1}{8r^2} &= -\frac{\mu r}{[r^2 + (1+\mu)^2 z^2]^{3/2}}, \\ rv'' + 2r'v' &= 0, \\ z'' &= -\frac{2m(1+\mu)z}{[r^2 + (1+\mu)^2 z^2]^{3/2}}. \end{aligned} \tag{5}$$

3. The equations of variation

For $\mu = 0$, the first two equations of (5) reduce to those of the two-body problem, and their solutions are*

$$\begin{aligned} r = r_0 &= \frac{a}{2}\left[1 - e\cos n(t-t_0) + \frac{e^2}{2}\{1 - \cos 2n(t-t_0)\} + \cdots\right], \\ v = v_0 &= n(t-t_0) + 2e\sin n(t-t_0) + \frac{5}{4}e^2\sin 2n(t-t_0) + \cdots. \end{aligned} \tag{6}$$

In these equations both a and n are unity, while t_0 can be chosen to be zero without loss of generality, but these numerical values are not substituted at present as we wish later to differentiate r_0 with respect to t_0.

The second equation of (5) admits the integral

$$r^2 v' = c. \tag{7}$$

When $\mu = 0$ the third equation of (5) becomes

$$z'' = -\frac{2mz}{[r^2 + z^2]^{3/2}}, \tag{8}$$

where r has the value r_0 obtained in (6). Now instead of putting $m = 1/2$ in the numerator of (8), as was done in the first two equations of (4), we make the substitution

$$m = m_0 + \lambda,$$

where m_0 is considered as a constant and λ as a variable parameter. For the purpose of computation we must substitute $\lambda = m - m_0$ in the final results.

* Moulton's *Celestial Mechanics*, pp. 154, 155.

It was shown in case II of my dissertation that periodic solutions of equation (8) exist, and are power series in $\lambda^{1/2}$ of the form

$$z_0 = \pm \lambda^{1/2}\left[\frac{-\sqrt{2}}{3\sqrt{m_0}}\sin \sigma n (t - t_0) + \text{power series in } e\right] + \cdots$$

$$= \pm \sum_{j=0}^{\infty} z_{2j+1} (\lambda^{1/2})^{2j+1}, \tag{9}$$

where each z_{2j+1} is a power series in e of the form

$$z_{2j+1} = \sum_{\kappa=0}^{\infty}\left[\sum_{l=0}^{j}\sum_{t=0}^{\kappa} c_{\pm l, t, \kappa}^{2j+1} \sin n\{(2l+1)\sigma \pm i\}(t - t_0)\right] e^{\kappa}.$$

The $c_{\pm l, t, \kappa}^{2j+1}$ are real constants and σ is a power series in e of the form

$$\sigma = 4\sqrt{m_0} + \frac{3\sqrt{m_0}(1 - 16m_0)}{1 - 64m_0} e^2 + \cdots.$$

As power series similar to z_0 in (9) occur frequently in the sequel, we shall call them $\lambda^{1/2}$-*sine-series*.

Since σ is a continuous function of e, values of e can be chosen so that σ converges to a rational fraction N/ν, where N and ν are integers relatively prime. The period of the solutions (9) is then

$$T = 2\nu\pi = \frac{2N\pi}{\sigma}. \tag{10}$$

In § 171 of my dissertation,* a reference was given to show that the two solutions in (9) represent the same orbit if ν is even, but in the one the infinitesimal body is half a period ahead of its position in the other. If ν is odd the orbits for plus and minus are geometrically distinct. In the sequel we shall take only the plus sign in (9) as the solutions obtained from the minus sign are entirely analogous to those obtained when the plus sign is taken.

In order to find the solutions of (5) when μ is not zero, we make the substitutions

$$r = r_0 + p, \qquad c = c_0 + d\mu, \qquad z = z_0 + w, \tag{11}$$

where p and w vanish with μ, and d is an arbitrary constant. From (7) it follows that

$$c_0 = r_0^2 v_0'.$$

When v' is eliminated from the first equation of (5) by the integral (7), and the substitutions (11) are made, the terms in the first equation of (5) independent of μ are zero. Hence

$$c_0^2 = r_0'' r_0^3 + \tfrac{1}{8} r_0. \tag{12}$$

* Moulton's *Periodic Orbits*.

The differential equations determining p and w then become

$$\begin{aligned} p'' + P(e,t)\,p &= P_0 + P_1\mu + P_2\mu^2 + \cdots, \\ w'' + W(\lambda,e,t)\,w &= W_0 + W_1\mu + W_2\mu^2 + \cdots, \end{aligned} \tag{13}$$

where P, W, P_j, W_j $(j = 0, 1, 2, \cdots)$ have the following properties:

(1) P is a power series in e whose coefficients are sums of cosines of multiples of t. In so far as the computations have been made, we have

$$P(e,t) = 1 + 6e\cos t + e^2(3 + 12\cos 2t) + \cdots.$$

(2) W is a power series in λ whose coefficients are power series in e. The coefficients of the various powers of e have the form

$$\sum_{k,l} c_{k,l}\cos\,(2k\sigma \pm l)t,$$

where k and l are positive integers or zero, and $c_{k,l}$ are real constants. As power series of this kind occur frequently in the sequel we shall call them *λ-cosine-series*.

(3) P_0 is a power series in p beginning with the term in p^2. The coefficients of the various powers of p are power series in e similar in form to P.

(4) P_j $(j = 1, 2, 3, \cdots)$ are power series in p and w. The coefficients of the terms in which w occurs in even degrees are λ-cosine-series. The coefficients of the terms in which w occurs in odd degrees are $\lambda^{1/2}$-sine-series.

(5) W_j $(j = 0, 1, 2, \cdots)$ are power series in p and w. The coefficients of the various powers of p and w are $\lambda^{1/2}$-sine-series if w occurs in even degrees, and λ-cosine-series if w occurs in odd degrees.

If we put the left members in (13) equal to zero, we obtain the equations of variation, which are

$$(a)\quad p'' + P(e,t)\,p = 0, \qquad (b)\quad w'' + W(\lambda,e,t)\,w = 0. \tag{14}$$

4. The solutions of the equations of variation

The generating solutions of the equations of variation are r_0 and z_0 in (6) and (9) respectively. Each of these two generating solutions contains the arbitrary constant t_0, and hence one set of solutions of (14) is obtained by taking the first partial derivatives of r_0 and z_0 with respect to t_0.* One solution of the first equation of (14) is therefore

$$\frac{\partial r_0}{\partial t_0} = -\frac{e}{2}[\sin t + e\sin 2t + \cdots] = -\frac{e}{2}\psi(e,t), \tag{15}$$

and as we multiply this solution later by an arbitrary constant, we may disregard the factor $-e/2$.

* Poincaré's *Les Méthodes Nouvelles de la Mécanique Céleste*, vol. 1, Chap. IV. Moulton's *Periodic Orbits*, §§ 32, 33.

When the generating solution contains an arbitrary constant one of the characteristic exponents of the solutions of the equations of variation is zero, and since the sum of the characteristic exponents is zero both these exponents are zero. In this case one of the solutions is a periodic function while the other is the sum of a periodic function and t times the former solution. That is, the second solution of (14a) is

$$p = \theta + At\psi,$$

where θ and A are power series in e, the former having periodic coefficients, the latter having constant coefficients. In order to determine θ and A we substitute $p = \theta + At\psi$ in (14a) and integrate as power series in e. We choose the initial values $\theta(0) = 1$, $\theta'(0) = 0$ and determine A so that θ shall be periodic. The computation gives

$$\theta = \cos t - e(3 - 2\cos t - \cos 2t)$$
$$- e^2(6 - \tfrac{23}{8}\cos t - 2\cos 2t - \tfrac{9}{8}\cos 3t) + \cdots,$$
$$A = \gamma e^2 = [3 + \text{power series in } e]e^2.$$

Hence the two solutions of (14a) are ψ and $\theta + \gamma e^2 t\psi$. The determinant of these two solutions is a constant,* and its value at $t = 0$ is

$$D = -\theta(0)\psi'(0) = -(1 + 2e + \cdots). \tag{16}$$

This determinant is different from zero for $e = 0$ and therefore remains different from zero for $|e|$ sufficiently small. Consequently the two solutions (15) constitute a fundamental set and the general solution of the first equation of (14) is

$$p = A_1\psi + A_2[\theta + \gamma e^2 t\psi], \tag{17}$$

where A_1 and A_2 are the constants of integration. The functions ψ and θ are periodic with the period 2π and *à posteriori* with the period T.

Similarly, one of the solutions of the second equation of (14) is

$$\frac{\partial z_0}{\partial t_0} = \lambda^{1/2}\phi(\lambda, e, t), \tag{18}$$

where ϕ is a power series in λ similar to a λ-cosine-series except that it has odd multiples of σ in the arguments of the cosines instead of even multiples. By computation it is found that

$$\phi(0) = \frac{\sqrt{2}\sigma}{3\sqrt{m_0}} + \cdots, \qquad \phi'(0) = 0. \tag{19}$$

The second solution of (14b) could be found in the same way as the second

* Moulton's *Periodic Orbits*, § 18.

solution of (14*a*), but we shall determine it by a shorter method. We may consider λ as an arbitrary constant and in this case the other solution of (14*b*) is

$$\frac{\partial z_0}{\partial \lambda} = \left(\frac{\partial z_0}{\partial \lambda}\right) + \frac{\partial z_0}{\partial n}\frac{\partial n}{\partial \lambda}, \tag{20}$$

where $(\partial z_0/\partial\lambda)$ denotes that the differentiation is performed only in so far as λ enters z_0 explicitly. Hence

$$\left(\frac{\partial z_0}{\partial \lambda}\right) = \frac{\partial z_0}{\partial (\lambda^{1/2})}\frac{\partial (\lambda^{1/2})}{\partial \lambda} = -\frac{1}{\lambda^{1/2}}\chi,$$

where χ is a power series in λ with coefficients which are similar in form to the coefficients in a $\lambda^{1/2}$-sine-series. The values of $\chi(0)$ and $\chi'(0)$ are found by computation to be

$$\chi(0) = 0, \qquad \chi'(0) = \left[\frac{\sqrt{2}\sigma}{6\sqrt{m_0}} + \text{power series in } \lambda\right]. \tag{21}$$

The partial derivative $\partial z_0/\partial n$ has the value

$$\frac{\partial z_0}{\partial n} = -\lambda^{1/2} t\phi,$$

and since

$$n = \frac{k\sqrt{2m}}{a^{3/2}} = \frac{k\sqrt{2(m_0 + \lambda)}}{a^{3/2}},$$

then $\partial n/\partial\lambda = 1$. Therefore the second solution of (14*b*) is

$$-\frac{1}{\lambda^{1/2}}[\chi + \lambda t\phi].$$

The two solutions of (14*b*) then take the form

$$\lambda^{1/2}\phi \quad \text{and} \quad -\frac{1}{\lambda^{1/2}}[\chi + \lambda t\phi], \tag{22}$$

where ϕ and χ are periodic with the period T. As in the solution of (14*a*) we can omit the constants $\lambda^{1/2}$ and $-\lambda^{-1/2}$, and consider ϕ and $\chi + \lambda t\phi$ as the two solutions. The determinant of these solutions is a constant,* and at $t = 0$ the value is

$$\Delta = \phi(0)[\chi'(0) + \lambda\phi(0)] = \frac{\sigma^2}{9m_0} + \cdots, \tag{23}$$

a power series in λ in which the coefficients are power series in e. The deter-

* Moulton's *Periodic Orbits*, § 18.

minant Δ is different from zero for $e = \lambda = 0$, and therefore remains different from zero for $|e|$ and $|\lambda|$ sufficiently small. Consequently the two solutions ϕ and $\chi + \lambda t\phi$ constitute a fundamental set and the general solution of (14b) is

$$w = B_1 \phi + B_2 [\chi + \lambda t\phi], \tag{24}$$

where B_1 and B_2 are the constants of integration.

5. Existence of symmetrical periodic orbits

Let us take the initial conditions

$$p(0) = \alpha_1, \quad p'(0) = 0, \quad w(0) = 0, \quad w'(0) = \alpha_2. \tag{25}$$

With these initial conditions it can be shown from the properties of the differential equations (13) that p is even in t and w is odd in t. Hence sufficient conditions that p and w shall be periodic with the period T are

$$p'\left(\frac{T}{2}\right) = 0, \quad w\left(\frac{T}{2}\right) = 0. \tag{26}$$

In order to establish the existence of symmetrical periodic orbits, we integrate (13) as power series in α_1, α_2, and μ, subject to the initial conditions (25), and impose the periodicity conditions (26) upon the solutions. It will be shown that (26) can be satisfied by a determination of α_1 and α_2 as power series in μ.

The differential equations from which the linear terms in α_1 and α_2 are obtained are the same as the equations of variation. The solutions of these equations which satisfy the initial conditions (25) are

$$p = \frac{\alpha_1}{\theta(0)}[\theta + \gamma e^{\varrho t}\psi], \qquad w = \frac{\alpha_2 [\chi + \lambda t\phi]}{[\chi'(0) + \lambda\phi(0)]}.$$

When the terms in μ and higher degree terms in μ, α_1, and α_2 are taken in (13), the solutions satisfying (25) have the form

$$\begin{aligned} p &= \frac{\alpha_1}{\theta(0)}[\theta + \gamma e^{\varrho t}\psi] \\ &\quad + \text{terms in } \mu \text{ and higher degree terms in } \mu, \alpha_1, \text{ and } \alpha_2, \\ w &= \frac{\alpha_2 [\chi + \lambda t\phi]}{[\chi'(0) + \lambda\phi(0)]} \\ &\quad + \text{terms in } \mu \text{ and higher degree terms in } \mu, \alpha_1, \text{ and } \alpha_2. \end{aligned} \tag{27}$$

After imposing the conditions (26) upon (27), we obtain

$$0 = \frac{\alpha_1}{\theta(0)}\left[\theta'\left(\frac{T}{2}\right) + \gamma e^2 \frac{T}{2}\psi'\left(\frac{T}{2}\right) + \gamma e^2 \psi\left(\frac{T}{2}\right)\right]$$

$$+ \text{terms in } \mu \text{ and higher degree terms in } \mu, \alpha_1, \text{ and } \alpha_2,$$

(28)

$$0 = \frac{\alpha_2\left[\chi\left(\frac{T}{2}\right) + \lambda\frac{T}{2}\phi\left(\frac{T}{2}\right)\right]}{\chi' + \lambda\phi(0)}$$

$$+ \text{terms in } \mu \text{ and higher degree terms in } \mu, \alpha_1, \text{ and } \alpha_2.$$

From the form of ψ, θ and χ it follows that

$$\psi\left(\frac{T}{2}\right) = \theta'\left(\frac{T}{2}\right) = \chi\left(\frac{T}{2}\right) = 0,$$

and therefore the determinant of the linear terms in α_1 and α_2 in (28)

(29)

$$\delta = -\frac{\gamma e^2 \lambda T^2 \psi'\left(\frac{T}{2}\right)\phi\left(\frac{T}{2}\right)\psi'(0)\phi(0)}{4D\Delta}.$$

The value of δ depends upon the parity of ν and N in $T = 2\nu\pi = 2N\pi/\sigma$. The determinants D and Δ are both different from zero for $|e|$ and $|\lambda|$ sufficiently small and are both finite. Hence δ can become zero only when the numerator vanishes. We shall show that the numerator of (29) is different from zero provided that e and λ are not zero but sufficiently small numerically.

Suppose ν is even, then N is odd, since ν and N are relatively prime, and from the form of ψ and ϕ it follows that

$$\psi'\left(\frac{T}{2}\right) = \psi'(0) \neq 0, \qquad \phi\left(\frac{T}{2}\right) = -\phi(0) \neq 0.$$

If ν is odd then

$$\psi'\left(\frac{T}{2}\right) = -1 + 2e + \cdots,$$

and

$$\phi\left(\frac{T}{2}\right) = \frac{\sqrt{2}}{3\sqrt{m_0}} + e\,[\text{power series in } e] \quad \text{or} \quad -\frac{\sqrt{2}}{3\sqrt{m_0}} + e\,[\text{power series in } e],$$

according as N is even or odd. Therefore δ is different from zero whatever be the parity of ν and N provided $|e|$ and $|\lambda|$ are sufficiently small. Consequently equations (28) can be solved for α_1 and α_2 as power series in μ which vanish with μ and converge for $|\mu|$ sufficiently small. Since the initial

values of p and w are power series in μ, the solutions will have the form

$$p = \sum_{i=1}^{\infty} p_i \mu^i, \qquad w = \sum_{i=1}^{\infty} w_i \mu^i, \tag{30}$$

where each p_i and w_i is periodic with the period T. The constant d which appears first in (11) enters these solutions as an arbitrary, but when a value is assigned to d the solutions in (30) are unique.

6. Direct construction of the symmetrical periodic orbits

Let us substitute (30) in (13) and equate the coefficients of the various powers of μ. We thus obtain a series of differential equations which can be integrated step by step. The constants of integration appearing at each stage are to be determined so that the solutions shall be periodic and satisfy the symmetrical initial conditions

$$p'(0) = 0, \qquad w(0) = 0.$$

When these symmetrical initial conditions are imposed on (30) we obtain

$$p_i'(0) = 0, \qquad w_i(0) = 0 \qquad (i = 1, \cdots \infty). \tag{31}$$

The differential equations for the terms in μ are

$$(a)\quad p_1'' + P(e,t)\,p_1 = P^{(1)}, \qquad (b)\quad w_1'' + W(\lambda, e, t)\,w_1 = W^{(1)}. \tag{32}$$

The left members of these equations are the same as the equations of variation. The right member $P^{(1)}$ is a λ-cosine-series and is independent of w_1. The right member of (32b) depends on p_1. The terms of $W^{(1)}$ which are independent of p_1 form a $\lambda^{1/2}$-sine-series and the remaining terms are also a $\lambda^{1/2}$-sine-series multiplied by p_1.

The complementary function of (32a) is

$$p_1 = a_1^{(1)}\psi + a_2^{(1)}[\theta + \gamma e^2 t\psi], \tag{33}$$

where $a_1^{(1)}$ and $a_2^{(1)}$ are the constants of integration. To obtain the particular integral we employ the method of the variation of parameters. By this method we have

$$\begin{aligned} (a_1^{(1)})'\,\psi + (a_2^{(1)})'\,[\theta + \gamma e^2 t\psi] &= 0, \\ (a_1^{(1)})'\,\psi' + (a_2^{(1)})'\,[\theta + \gamma e^2 (t\psi' + \psi)] &= P^{(1)}. \end{aligned} \tag{34}$$

The determinant of the coefficients of $(a_1^{(1)})'$ and $(a_2^{(1)})'$ is D, the same as in (16), and is different from zero. Hence

$$(a_1^{(1)})' = -\frac{[\theta + \gamma e^2 t\psi]\,P^{(1)}}{D}, \qquad (a_2^{(1)})' = \frac{\psi P^{(1)}}{D}. \tag{35}$$

When these equations are integrated and the results are substituted in (33), we obtain the solution

$$(36)\quad p_1 = A_1^{(1)}\psi + A_2^{(1)}[\theta + \gamma e^2 t\psi] + C^{(1)} - t\psi\,[L_0^{(1)} + \lambda^{\nu/2} e^N L_1^{(1)} \quad \text{or} \quad \lambda^{\nu} e^{2N} L_2^{(1)}],$$

according as ν is even or odd respectively. The terms in (36) not already defined have the following properties: $A_1^{(1)}$ and $A_2^{(1)}$ are the constants of integration; $C^{(1)}$ is a λ-cosine-series; $L_1^{(1)}$ and $L_2^{(1)}$ are power series in λ with coefficients which are constant power series in e, and $L_0^{(1)}$ is a power series in e with constant coefficients. In order that p_1 shall satisfy (31) and shall be periodic, the constants of integration must have the values

$$(37)\quad A_1^{(1)} = 0, \qquad A_2^{(1)} = \frac{1}{\gamma e^2} L_0^{(1)} + \frac{1}{\gamma}[\lambda^{\nu/2} e^{N-2} L_1^{(1)} \quad \text{or} \quad \lambda^{\nu} e^{2N-2} L_2^{(1)}],$$

according as ν is even or odd. Hence the solution of (32a) which is periodic and satisfies (31) is

$$(38)\qquad p_1 = \frac{1}{e^2} C_1(t),$$

where $C_1(t)$ is a λ-cosine-series.

With p_1 determined as in (38), the right member of (32b) takes the form of a $\lambda^{1/2}$-sine-series multiplied by $1/e^2$. The complete solution of (32b) is obtained in the same way as (36) was obtained. The complementary function of (32b) is

$$(39)\qquad w_1 = b_1^{(1)}\phi + b_2^{(1)}[\chi + \lambda t\phi],$$

and the equations analogous to (34) are

$$(b_1^{(1)})'\phi + (b_2^{(1)})'[\chi + \lambda t\phi] = 0,$$

$$(b_1^{(1)})'\phi' + (b_2^{(1)})'[\chi' + \lambda(t\phi' + \phi)] = W^{(1)}.$$

The determinant of the coefficients of $(b_1^{(1)})'$ and $(b_2^{(1)})'$ is Δ, the same as in (23), and is different from zero. Hence

$$(40)\qquad (b_1^{(1)})' = -\frac{[\chi + \lambda t\phi]\,W^{(1)}}{\Delta}, \qquad (b_2^{(1)})' = \frac{\phi W^{(1)}}{\Delta}.$$

When these equations are integrated and the results are substituted in (39), the general solution of (32b) is found to be

$$(41)\qquad w_1 = B_1^{(1)}\phi + B_2^{(1)}[\chi + \lambda t\phi] + \frac{1}{e^2} S^{(1)} - \frac{\lambda^{1/2}}{e^2} M^{(1)} t\phi,$$

where $B_1^{(1)}$ and $B_2^{(1)}$ are the constants of integration; $S^{(1)}$ is a $\lambda^{1/2}$-sine-series, and $M^{(1)}$ has the same form as $L_1^{(1)}$ or $L_2^{(1)}$ in (36). In order that w_1 shall satisfy the initial conditions (31) and be periodic, the constants of integration

must have the values

$$B_1^{(1)} = 0, \qquad B_2^{(1)} = \frac{M^{(1)}}{e^2\lambda^{1/2}}. \tag{42}$$

Hence the desired solution of (32b) is

$$w_1 = \frac{1}{e^2\lambda} S_1(t), \tag{43}$$

where $S_1(t)$ is a $\lambda^{1/2}$-sine-series.

The remaining steps of the integration can be carried on in the same way, and the solutions are almost entirely similar to those obtained at the first step. The presence of the terms in $1/e^2$ and $1/e^2\lambda$ in (38) and (43) introduces some slight changes in the forms of the solutions, and it is necessary to consider the differential equations up to the terms in μ^3 in order to discover the law according to which the remaining solutions proceed.

The terms in μ^2 yield the differential equations

$$(a)\quad p_2'' + P(e,t)p_2 = P^{(2)}, \qquad (b)\quad w_2'' + W(\lambda, e, t)w_2 = W^{(2)}. \tag{44}$$

The right member of (44a) is a λ-cosine-series multiplied by $1/e$, and the complete solution of the equation is

$$p_2 = A_1^{(1)}\psi + A_2^{(2)}[\theta + \gamma e^2 t\psi] + \frac{1}{e^2}C^{(2)} - \frac{1}{e^2}L^{(2)}t\psi, \tag{45}$$

where $A_1^{(2)}$ and $A_2^{(2)}$ are the constants of integration; $C^{(2)}$ is a λ-cosine-series; and $L^{(2)}$ is a power series in λ with coefficients which are constant power series in e. The numerical values of the coefficients in $L^{(2)}$ vary with the parity of ν, but the series has the same form whether ν is even or odd. In order that p_2 shall be periodic and satisfy the conditions (31) the constants of integration must have the values

$$A_1^{(2)} = 0, \qquad A_2^{(2)} = \frac{L^{(2)}}{e^4}, \tag{46}$$

and the desired solution for p_2 then takes the form

$$p_2 = \frac{1}{e^4}C_2(t),$$

where $C_2(t)$ is a λ-cosine-series.

The right member of (44b) is completely known when p_2 has been determined and is a $\lambda^{1/2}$-sine-series multiplied by $1/e^4\lambda$. The complete solution of (44b) is readily found to be

$$w_2 = B_1^{(2)}\phi + B_2^{(2)}[\chi + \lambda t\phi] + \frac{1}{e^4\lambda}S^{(2)} - \frac{1}{e^4\lambda^{1/2}}M^{(2)}t\phi, \tag{47}$$

where the undefined terms in (47) denote functions similar to those represented by the corresponding terms in (41). If the initial and periodicity conditions

are to be satisfied then

$$B_1^{(2)} = 0, \qquad B_2^{(2)} = \frac{M^{(2)}}{e^4 \lambda^{3/2}}.$$

Hence the desired solution for w_2 is

$$w_2 = \frac{1}{e^4 \lambda^2} S_2(t),$$

where $S_2(t)$ is a $\lambda^{1/2}$-sine-series.

The differential equations obtained from the terms in μ^3 are

$$(48)\quad (a)\quad p_3'' + P(e,t)\, p_3 = P^{(3)}, \qquad (b)\quad w_3'' + W(\lambda, e, t)\, w_3 = W^{(3)}.$$

The right member $P^{(3)}$ is a λ-cosine-series multiplied by $1/e^4\lambda$, and the complete solution of (48a) is

$$(49)\qquad p_3 = A_1^{(3)} \psi + A_2^{(3)} [\theta + \gamma e^2 t\psi] + \frac{1}{e^4 \lambda} C^{(3)} - \frac{1}{e^4 \lambda} L^{(3)} t\psi,$$

where the undefined terms denote functions similar to those represented by the corresponding terms in (45). If the initial and periodicity conditions are to be satisfied, then

$$A_1^{(3)} = 0, \qquad A_2^{(3)} = \frac{1}{\gamma e^6 \lambda} L^{(3)},$$

and the desired solution for p_3 becomes

$$p_3 = \frac{1}{e^6 \lambda} C_3(t),$$

where $C_3(t)$ is a λ-cosine-series.

From the forms of p_3 and w_2 it follows that $W^{(3)}$ is a $\lambda^{1/2}$-sine-series multiplied by $1/e^6\lambda^2$. Hence the solution for w_3 will be a $\lambda^{1/2}$-sine-series multiplied by $1/e^6\lambda^3$. By an induction to the general term it can readily be shown that p_j and w_j have the forms

$$p_j = \frac{1}{e^{2j} \lambda^{j-2}} C_j(t) \qquad (j = 2, 3, \cdots \infty),$$

$$w_j = \frac{1}{e^{2j} \lambda^j} S_j(t) \qquad (j = 1, 2, \cdots \infty),$$

where each $C_j(t)$ is a λ-cosine-series and each $S_j(t)$ is a $\lambda^{1/2}$-sine-series.

7. Proof that all periodic orbits represented by power series in μ are symmetrical

The usual method of proving that all periodic orbits are symmetrical is to assume the initial conditions

$$(49)\quad p(0) = \alpha_1, \qquad p'(0) = \alpha_2, \qquad w(0) = \alpha_3, \qquad w'(0) = 0,$$

and integrate equations (13) as power series in α_i $(i = 1, 2, 3)$ and μ. Sufficient conditions that the solutions shall be periodic are

$$p(T) - p(0) = 0, \quad p'(T) - p'(0) = 0, \quad w(T) - w(0) = 0, \quad w'(T) - w'(0) = 0. \tag{50}$$

If there exists an integral of the differential equations (5) it could be shown by means of this integral that the last condition of (50) would be reduntant. Then the three independent conditions of (50) could be satisfied by a proper determination of α_i $(i = 1, 2, 3)$ as power series in μ provided a certain determinant did not vanish. Thus periodic solutions would be shown to exist and they would be unique for any assigned value of d which would enter the solutions through (11). Since the symmetrical solutions are also unique under the same circumstance and since the general orbits having the initial conditions (49) would include those which are symmetrical, all the solutions would therefore be symmetrical.

In the problem under consideration no integral of equations (5) is known to exist and we can not use the foregoing method. We show, however, from the construction of the solutions that the symmetrical initial conditions are a consequence of the periodicity conditions and therefore that all the orbits represented by solutions which are power series in μ with periodic coefficients are symmetrical.

If the symmetrical initial conditions (31) are not imposed upon (36), the periodic solution for p_1 is

$$p_1 = A_1^{(1)} \psi + C_1(t). \tag{51}$$

When this value of p_1 is substituted in the right member of the differential equations defining w_1 we obtain

$$w_1'' + W(\lambda, e, t) w_1 = W^{(1)} + A_1^{(1)} \overline{W}^{(1)},$$

where $\overline{W}^{(1)}$ is similar to a $\lambda^{1/2}$-sine-series except that it has cosines instead of sines. The complete solution of this equation is the same as (41) with the additional terms

$$A_1^{(1)} [\text{periodic terms} + \text{terms in } t^2 \phi \text{ and } t\chi].$$

The only way in which this solution can be made periodic is by putting $A_1^{(1)} = 0$ and then determining $B_2^{(1)}$ as in (42). Hence the periodic solution for w_1 which does not satisfy the condition $w_1(0) = 0$ becomes

$$w_1 = B_1^{(1)} \phi + \frac{1}{e^2 \lambda} S_1(t),$$

where $B_1^{(1)}$ remains undetermined at this step. In order to obtain the value

of $B_1^{(1)}$ we must consider the differential equation which defines p_2. It has the form

$$p_2'' + P(e,t)\,p_2 = P^{(2)} + B_1^{(1)}\,\bar{P}^{(2)}, \tag{52}$$

where $\bar{P}^{(2)}$ is similar to a $\lambda^{1/2}$-sine-series except that it has even multiples of σ instead of odd. The complete solution of (52) is the same as (45) with the additional terms

$$B_1^{(1)}\,[\text{periodic terms} + \text{terms in } t^2\psi \text{ and } t\theta\,].$$

The only way in which p_2 can be made periodic is by putting $B_1^{(1)} = 0$ and then determining $A_2^{(2)}$ in the same way as in (46).

Thus the integrations of the differential equations arising at each step can be obtained in the same way and the process continued as far as desired. The constant of integration which is not determined by the condition that p_j shall be periodic must be zero in order that w_j shall be periodic; and similarly the constant of integration which is not determined by the condition that w_j shall be periodic must be zero in order that p_{j+1} shall be periodic. The solutions thus obtained are the same as those obtained for the symmetrical orbits. Hence the symmetrical initial conditions (31) are a consequence of the periodicity conditions and all periodic orbits represented by power series in μ are symmetrical.

8. Integration of equation (7)

There remains one differential equation which has not been integrated, viz.,

$$r^2\,v' = c. \tag{7}$$

In order to integrate this equation we make in (7) the substitutions

$$v = v_0 + q, \qquad r = r_0 + p, \qquad c = c_0 + d\mu,$$

where q vanishes with μ, and p has the solution obtained in § 6. Then equation (7) becomes

$$q' = Q_1\,\mu + Q_2\,\mu^2 + \cdots + Q_j\,\mu^j + \cdots, \tag{54}$$

where each Q_j is similar in form to p_j, that is, each Q_j is a λ-cosine-series multiplied by $1/e^{2j}\,\lambda^{j-2}$. As the right side of (54) contains constant terms, the integration will yield terms in t unless these terms can be made to vanish. Now the constant d appears in these terms but as the constant part of the coefficient of μ does not vanish when $d = 0$, it is impossible to determine d so as to eliminate the constant terms in (54). Hence the integration of (54) gives

$$q = \sum_{j=1}^{\infty} [\,q_j + t\bar{q}_j\,]\,\mu^j + q_0,$$

where each q_j is similar in form to Q_j except that it has sines instead of cosines, each $\bar{q}_j$ is similar to $Q_j(0)$, and q_0 is the constant of integration. Since the equal masses are started from the ζ-axis at the initial time then $v(0) = 0$. Since $v_0(0) = 0$ it follows that $q(0) = 0$ and as each $q_j(0) = 0$ then $q_0 = 0$.

This completes the integration of the differential equations defining the motion of the three finite bodies m_1, m_2, and μ. The constant of integration arising from the integral of areas remains undetermined, but the remaining constants of integration can be determined by the periodicity conditions alone. The mass μ is restricted in magnitude because of certain convergence conditions but it can be increased step by step by making the analytical continuation of the solutions already obtained, provided the series do not pass through any singularities in the intervals. This can be done by the method already developed.

QUEEN'S UNIVERSITY, KINGSTON, ONTARIO,
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SURFACES Ω AND THEIR TRANSFORMATIONS*

BY

LUTHER PFAHLER EISENHART

INTRODUCTION

In a recent memoir† we showed that a surface S referred to a conjugate system with equal point invariants admits of transformations into surfaces upon which the parametric curves are of the same kind such that, if S_1 is a transform, for the congruence (G) of lines joining corresponding points on S and S_1 the developables cut these surfaces in the parametric curves, and the focal points on any line of the congruence are harmonic with respect to the points of S and S_1 on the line. In this case we say that S and S_1 are in the relation of a *transformation* K. In the present memoir we consider the particular case when the congruence (G) is a normal one. Surfaces orthogonal to such a congruence have been considered by Demoulin‡ who called them *surfaces* Ω. He gives practically no consideration to the surfaces S and S_1 which are fundamental in the discussion of the present paper. They belong to a particular class of surfaces, which we call *surfaces* C. It is by the characterization of surfaces C and the establishment of transformations K of them that we arrive at a general theory of transformations of surfaces Ω.

As shown in § 2 surfaces C are characterized by the property that the parametric conjugate system with equal point invariants is $2, O$ in the sense of Guichard,§ the complementary function being the distance to a surface Ω normal to the congruence (G). From the general theory of systems $2, O$ it follows that the spheres whose centers lie on a surface C and whose radii are equal to the complementary function envelop two surfaces upon which the lines of curvature correspond, and these surfaces are said to be in the relation of a transformation of Ribaucour. Hence there are two congruences (G) associated with a surface C, and with each congruence another surface C, namely the surface S_1 referred to above, but which hereafter will be called the surfaces C_0 and C_0'. We say that C_0 and C_0' are the conjugate surfaces of C.

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† These Transactions, vol. 15 (1914), pp. 397–430. A reference to an equation of this memoir will be of the form M. (−).

‡ Comptes Rendus, vol. 153 (1911), pp. 590–593, 705–707, 927–929.

§ Annales de l'école normale supérieure, ser. 3, vol. 14 (1897), pp. 467–516.

The surface C and each of these surfaces are in the relation of a transformation K such that the congruence (G) is normal.

In § 6 we establish the existence of transformations K_m of surfaces C into surfaces of the same kind, the subscript m denoting an essential constant which enters in the equations of the transformation. We use the subscript only in connection with transformations K which transform a surface C into a surface C. Each of these transformations gives rise to a transformation of the Ribaucour type of surfaces Ω, which we call a *transformation* A_m.

In a previous paper we showed that transformations K admit a theorem of permutability. We generalize it in the present memoir so as to give eight surfaces interrelated by transformations K. By means of this result we establish a theorem of permutability for the transformations K_m of surfaces C and for the transformations A_m of surfaces Ω.

Demoulin* showed that with a given surface Ω there is associated a second surface Ω such that the two surfaces have the same spherical representation of their lines of curvature. It can be shown that this relation is a special case of the transformations A_m.

An isothermic surface is a surface C, but in this case the congruence (G) is normal to C and hence C coincides with the surface Ω determined by it. Now the transformations A_m are the well-known transformations D_m discovered by Darboux and investigated at length by Bianchi.†

In the closing section we consider the case where a surface C is the middle surface of the congruence (G), under which condition the surface C_0 is at infinity. Now the surfaces normal to (G) are surfaces with isothermic spherical representation of their lines of curvature. We have studied‡ the transformations of these surfaces in a former memoir and now join the earlier results with those of the present study.

When the Lie line-sphere transformation is applied to a surface Ω, the resulting surface possesses a conjugate system whose tangents form W-congruences. Surfaces of this sort have been studied by Tzitzéica,§ who called them *surfaces* R. A part of Demoulin's investigations have to do with surfaces R. We content ourselves here with the remark that the results of the present memoir establish the existence of transformations of surfaces R, such that a surface and its transform are the focal surfaces of a W-congruence.‖

In a subsequent memoir we shall put the equations of a transformation A_m in a different form and study certain types of surfaces Ω.

* Loc. cit., p. 928.

† For a full discussion of this case, see M., pp. 422–428.

‡ These Transactions, vol. 9 (1908), pp. 149–177.

§ Comptes Rendus, vol. 153 (1911), p. 1127.

‖ Cf. Tzitzéica, loc. cit.

1. Equations of general transformations K

If S is a surface referred to a conjugate system with equal point invariants, its cartesian coördinates x, y, z, are solutions of an equation of the form

$$\frac{\partial^2\theta}{\partial u\partial v}+\frac{\partial \log\sqrt{\rho}}{\partial v}\frac{\partial\theta}{\partial u}+\frac{\partial \log\sqrt{\rho}}{\partial u}\frac{\partial\theta}{\partial v}=0, \tag{1}$$

where in general ρ is a function of u and v. If θ_1 is solution of this equation, linearly independent of x, y, and z, and λ_1 is the function defined by

$$\frac{\partial\lambda_1}{\partial u}=-\rho\frac{\partial\theta_1}{\partial u},\qquad \frac{\partial\lambda_1}{\partial v}=\rho\frac{\partial\theta_1}{\partial v}, \tag{2}$$

the functions x_1, y_1, z_1, given by equations of the form

$$\begin{aligned}\frac{\partial x_1}{\partial u}&=\frac{\rho}{\lambda_1}\left[(x_1-x)\frac{\partial\theta_1}{\partial u}+\theta_1\frac{\partial x}{\partial u}\right],\\ \frac{\partial x_1}{\partial v}&=-\frac{\rho}{\lambda_1}\left[(x_1-x)\frac{\partial\theta_1}{\partial v}+\theta_1\frac{\partial x}{\partial v}\right],\end{aligned} \tag{3}$$

are solutions of the equation

$$\frac{\partial^2\theta'}{\partial u\partial v}+\frac{\partial \log\sqrt{\rho_1}}{\partial v}\frac{\partial\theta'}{\partial u}+\frac{\partial \log\sqrt{\rho_1}}{\partial u}\frac{\partial\theta'}{\partial v}=0, \tag{4}$$

where

$$\sqrt{\rho_1}=\lambda_1/\sqrt{\rho}\,\theta_1. \tag{5}$$

Evidently the parametric curves on S_1, whose cartesian coördinates are x_1, y_1, and z_1, form a conjugate system with equal point invariants. Moreover, we have shown* that for the congruence (G) of lines joining corresponding points on S and S_1 the latter points are harmonic with respect to the focal points. The coördinates of the focal points are of the form

$$\frac{\lambda_1 x_1-\rho\theta_1 x}{\lambda_1-\rho\theta_1},\qquad \frac{\lambda_1 x_1+\rho\theta_1 x}{\lambda_1+\rho\theta_1}. \tag{6}$$

If 2ω denotes the angle between the coördinate curves at a point of S, and if X_1, Y_1, Z_1 and X_2, Y_2, Z_2 denote the direction-cosines of the bisectors of the angles between the tangents to these curves at the point, we have

$$\frac{\partial x}{\partial u}=\sqrt{E}\,(\cos\omega\,X_1-\sin\omega\,X_2),\qquad \frac{\partial x}{\partial v}=\sqrt{G}\,(\cos\omega\,X_1+\sin\omega\,X_2), \tag{7}$$

where E, F, and G are the fundamental coefficients of S and

$$F=\sqrt{EG}\cos 2\omega. \tag{8}$$

* M., p. 400.

We introduce three functions a_1, b_1, w_1 by the equation

$$x_1 - x = \frac{1}{m_1 \lambda_1}(a_1 X_1 + b_1 X_2 + w_1 X), \tag{9}$$

where m_1 is an arbitrary constant, it being understood that similar equations hold for y_1 and z_1 with the same functions a_1, b_1, w_1.

When the above expression for x_1 is substituted in equations (3), we find that a_1, b_1, w_1 must satisfy the equations

$$\begin{aligned}
\frac{\partial a_1}{\partial u} &= -m_1(\lambda_1 - \rho\theta_1)\sqrt{E}\cos\omega + b_1 A + w_1 D/(2\sqrt{E}\cos\omega),\\
\frac{\partial a_1}{\partial v} &= -m_1(\lambda_1 + \rho\theta_1)\sqrt{G}\cos\omega - b_1 B + w_1 D''/(2\sqrt{G}\cos\omega),\\
\frac{\partial b_1}{\partial u} &= m_1(\lambda_1 - \rho\theta_1)\sqrt{E}\sin\omega - a_1 A - w_1 D/(2\sqrt{E}\sin\omega),\\
\frac{\partial b_1}{\partial v} &= -m_1(\lambda_1 + \rho\theta_1)\sqrt{G}\sin\omega + a_1 B + w_1 D''/(2\sqrt{G}\sin\omega),
\end{aligned} \tag{10}$$

$$\frac{\partial w_1}{\partial u} = -\frac{D}{2\sqrt{E}}\left(\frac{a_1}{\cos\omega} - \frac{b_1}{\sin\omega}\right), \qquad \frac{\partial w_1}{\partial v} = -\frac{D''}{2\sqrt{G}}\left(\frac{a_1}{\cos\omega} + \frac{b_1}{\sin\omega}\right),$$

where

$$A = \frac{\sqrt{E}}{\sqrt{G}}\frac{\partial \log\sqrt{\rho}}{\partial v}\sin 2\omega - \frac{\partial\omega}{\partial u}, \qquad B = \frac{\sqrt{G}}{\sqrt{E}}\frac{\partial \log\sqrt{\rho}}{\partial u}\sin 2\omega - \frac{\partial\omega}{\partial v}. \tag{11}$$

If we put for the sake of brevity

$$T_1^2 = a_1^2 + b_1^2 + w_1^2, \tag{12}$$

it follows from (10) that

$$\begin{aligned}
T_1\frac{\partial T_1}{\partial u} &= m_1(\rho\theta_1 - \lambda_1)\sqrt{E}(a_1\cos\omega - b_1\sin\omega),\\
T_1\frac{\partial T_1}{\partial v} &= -m_1(\rho\theta_1 + \lambda_1)\sqrt{G}(a_1\cos\omega + b_1\sin\omega).
\end{aligned} \tag{13}$$

2. Characterization of surfaces Ω

In the preceding section we denoted by (G) the congruence formed by the lines joining corresponding points on a surface S and on a surface arising from S by a transformation K. We consider now the case when such a congruence (G) is a normal. We shall find that the surface S is not a general one. Any surface which leads to such a normal congruence we call a *surface C*. We assume that such a surface is known and we denote by C_0 the conjugate surface which with C determines the congruence. We indicate by a subscript

zero functions belonging to C_0 and also the functions which define the transformation K from C into C_0. Furthermore, we say that C_0 is in the relation of a *transformation* K_0 to C.

From (9) and (12) it follows that the distance between corresponding points M and M_0 on C and C_0 is $T_0/\lambda_0 m_0$. Hence the coördinates $\bar{x}$, $\bar{y}$, $\bar{z}$, of a point on the line MM_0 are of the form

$$\bar{x} = x + \frac{t\lambda_0 m_0}{T_0}(x_0 - x). \tag{14}$$

In order that this point describe a surface $\bar{S}$ normal to the lines of the congruence (G), it is necessary and sufficient that

$$\sum (x_0 - x)\frac{\partial \bar{x}}{\partial u} = 0, \qquad \sum (x_0 - x)\frac{\partial \bar{x}}{\partial v} = 0.$$

In consequence of equations (13) these conditions lead to

$$\begin{aligned} &\frac{\partial t}{\partial u} + \frac{\sqrt{E}}{T_0}(\cos\omega\, a_0 - \sin\omega\, b_0) = 0, \\ &\frac{\partial t}{\partial v} + \frac{\sqrt{G}}{T_0}(\cos\omega\, a_0 + \sin\omega\, b_0) = 0. \end{aligned} \tag{15}$$

If these equations be differentiated with respect to v and u respectively, we have

$$C + (\rho\theta_0 + \lambda_0)D = 0, \qquad C - (\rho\theta_0 - \lambda_0)D = 0, \tag{16}$$

where we have put for the sake of brevity

$$C = \frac{\partial^2 t}{\partial u \partial v} + \frac{\partial \log\sqrt{\rho}}{\partial v}\frac{\partial t}{\partial u} + \frac{\partial \log\sqrt{\rho}}{\partial u}\frac{\partial t}{\partial v},$$

$$D = m_0\sqrt{EG}\,(a_0^2\cos^2\omega - b_0^2\sin^2\omega - T_0^2\cos 2\omega T_0^3).$$

From the form of (16) it follows that C and D must be zero. The first shows that t is a solution of equation (1) and the second necessitates

$$a_0^2\cos^2\omega - b_0^2\sin^2\omega - T_0^2\cos 2\omega = 0. \tag{17}$$

In consequence of (12) this equation may be written in the form

$$(b_0^2 + w_0^2)\cos^2\omega - (a_0^2 + w_0^2)\sin^2\omega = 0. \tag{17'}$$

From equations (8), (15), and (17) it follows that

$$\frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial t}{\partial u}\frac{\partial t}{\partial v} = 0. \tag{18}$$

Hence $x^2 + y^2 + z^2 - t^2$ is a solution of equation (1) as well as t. Adopting the notation of Guichard,* we have that the parametric conjugate system on C is $2, O$ the complementary function being t.

It is a well-known fact† that when a conjugate system $2, O$ is known on any surface Σ, the spheres whose centers lie on Σ and whose radii are equal to the complementary function are enveloped by two surfaces, upon which the lines of curvature correspond to the given conjugate system on Σ. Moreover, this is a characteristic property of envelopes of spheres with lines of curvature in correspondence. It is our purpose to show that when the system $2, O$ has equal point invariants the two envelopes are surfaces Ω.

Let $\bar{x}$, $\bar{y}$, $\bar{z}$ denote the cartesian coördinates of a surface $\overline{S}$ referred to its lines of curvature, $\overline{X}$, $\overline{Y}$, $\overline{Z}$ the direction-cosines of its normal, ρ_1, ρ_2 its principal radii of curvature. The point whose cartesian coördinates x, y, z are defined by

$$x = \bar{x} - \frac{\alpha}{\beta}\overline{X}, \qquad y = \bar{y} - \frac{\alpha}{\beta}\overline{Y}, \qquad z = \bar{z} - \frac{\alpha}{\beta}\overline{Z}, \tag{19}$$

describes a surface S upon which the parametric system is conjugate and $2, O$, provided that α and β are functions of u and v satisfying the equations§

$$\frac{\partial \alpha}{\partial u} + \rho_1 \frac{\partial \beta}{\partial u} = 0, \qquad \frac{\partial \alpha}{\partial v} + \rho_2 \frac{\partial \beta}{\partial v} = 0. \tag{20}$$

In this case $\overline{S}$ is one of the sheets of the envelope of the spheres with centers on S and of radius α/β.

The functions for $\overline{S}$ satisfy

$$\frac{\partial \bar{x}}{\partial u} + \rho_1 \frac{\partial \overline{X}}{\partial u} = 0, \qquad \frac{\partial \bar{x}}{\partial v} + \rho_2 \frac{\partial \overline{X}}{\partial v} = 0, \tag{21}$$

and similar equations in $\bar{y}$ and $\bar{z}$. Moreover, $\bar{x}$, $\bar{y}$, and $\bar{z}$ are solutions of the equation

$$\frac{\partial^2 \bar{\theta}}{\partial u \partial v} = \frac{\partial}{\partial v} \log \sqrt{\bar{E}} \frac{\partial \bar{\theta}}{\partial u} + \frac{\partial}{\partial u} \log \sqrt{\bar{G}} \frac{\partial \bar{\theta}}{\partial v}, \tag{22}$$

where $\bar{E}$ and $\bar{G}$ are the first fundamental coefficients of $\overline{S}$.

With the aid of (21) we obtain from (19) by differentiation

$$\begin{aligned} \frac{\partial x}{\partial u} &= \left(1 + \frac{\alpha}{\beta\rho_1}\right)\left(\frac{\partial \bar{x}}{\partial u} - \frac{1}{\beta}\frac{\partial \alpha}{\partial u}\overline{X}\right), \\ \frac{\partial x}{\partial v} &= \left(1 + \frac{\alpha}{\beta\rho_2}\right)\left(\frac{\partial \bar{x}}{\partial v} - \frac{1}{\beta}\frac{\partial \alpha}{\partial v}\overline{X}\right). \end{aligned} \tag{23}$$

*Annales de l'école normale supérieure, ser. 3, vol. 14 (1897), pp. 467–516.

† Cf. Darboux, *Leçons sur la théorie générale des surfaces*, vol. II, Paris, 1889, p. 326.

§ Cf. Darboux, l. c., p. 339.

The Codazzi equations for $\bar{S}$ may be written*

$$(24)\qquad \frac{\partial}{\partial v}\left(\frac{1}{\rho_1}\right)=\left(\frac{1}{\rho_2}-\frac{1}{\rho_1}\right)\frac{\partial \log\sqrt{\bar{E}}}{\partial v},\qquad \frac{\partial}{\partial u}\left(\frac{1}{\rho_2}\right)=\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)\frac{\partial \log\sqrt{\bar{G}}}{\partial u}.$$

From (20) and (24) we have

$$(25)\qquad \begin{aligned}\frac{\partial}{\partial u}\left(\frac{\alpha}{\beta\rho_2}\right)&=\left(1+\frac{\alpha}{\beta\rho_1}\right)\frac{1}{\rho_2\,\beta}\frac{\partial\alpha}{\partial u}+\frac{\alpha}{\beta}\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)\frac{\partial \log\sqrt{\bar{G}}}{\partial u},\\ \frac{\partial}{\partial v}\left(\frac{\alpha}{\beta\rho_1}\right)&=\left(1+\frac{\alpha}{\beta\rho_2}\right)\frac{1}{\beta\rho_1}\frac{\partial\alpha}{\partial v}+\frac{\alpha}{\beta}\left(\frac{1}{\rho_2}-\frac{1}{\rho_1}\right)\frac{\partial \log\sqrt{\bar{E}}}{\partial v}.\end{aligned}$$

Making use of these formulas, we obtain from (23) by differentiation and reduction

$$(26)\qquad \frac{\partial^2 x}{\partial u\partial v}=\frac{\partial}{\partial v}\log\left[\sqrt{\bar{E}}\left(1+\frac{\alpha}{\beta\rho_1}\right)\right]\frac{\partial x}{\partial u}+\frac{\partial}{\partial u}\log\left[\sqrt{\bar{G}}\left(1+\frac{\alpha}{\beta\rho_2}\right)\right]\frac{\partial x}{\partial v}.$$

Hence a necessary and sufficient condition that S have equal point invariants is

$$(27)\qquad \frac{\partial^2}{\partial u\partial v}\log\left(\frac{\sqrt{\bar{E}}}{\sqrt{\bar{G}}}\frac{1+\alpha/\beta\rho_1}{1+\alpha/\beta\rho_2}\right)=0.$$

It is our purpose to show that when this condition is satisfied, the point M' harmonic to M with respect to the centers of principal curvature of $\bar{S}$ describes a conjugate system with equal point invariants, and that consequently $\bar{S}$ is a surface Ω.

If we put

$$(28)\qquad x'=\bar{x}-\frac{\alpha'}{\beta'}X,\qquad y'=\bar{y}-\frac{\alpha'}{\beta'}Y,\qquad z'=\bar{z}-\frac{\alpha'}{\beta'}Z,$$

the necessary and sufficient condition that the point $M'(x', y', z')$ is the harmonic of M with respect to the centers of principal curvature of $\bar{S}$, the coördinates of these centers being of the form

$$(29)\qquad \bar{x}+\rho_1 X,\qquad \bar{x}+\rho_2 \bar{X},$$

is that α' and β' satisfy the condition

$$(30)\qquad \left(\frac{\alpha}{\beta}+\rho_1\right)\left(\frac{\alpha'}{\beta'}+\rho_2\right)+\left(\frac{\alpha}{\beta}+\rho_2\right)\left(\frac{\alpha'}{\beta'}+\rho_1\right)=0.$$

* E., pp. 157, 181. A reference of this sort is to the author's *Differential Geometry*, Ginn and Co., Boston, 1909.

In conformity with equation (30) we define a function σ by the equations

$$\begin{aligned}\beta' &= \sigma\left[\frac{1}{\rho_1}\left(1+\frac{\alpha}{\beta\rho_2}\right)+\frac{1}{\rho_2}\left(1+\frac{\alpha}{\beta\rho_1}\right)\right],\\ \alpha' &= -\sigma\left[\left(1+\frac{\alpha}{\beta\rho_1}\right)+\left(1+\frac{\alpha}{\beta\rho_2}\right)\right].\end{aligned} \tag{31}$$

In order that the point M' shall describe a conjugate system $2, O$, it is necessary and sufficient that α' and β' be a pair of solutions of equations (20). Expressing this condition, we find that σ must satisfy the equations

$$\begin{aligned}\frac{\partial \log\sigma}{\partial u} &= \frac{\alpha+\beta\rho_2}{(\rho_1-\rho_2)(\alpha+\beta\rho_1)}\frac{\partial\rho_1}{\partial u}+\frac{\partial}{\partial u}\log\frac{\beta\rho_1\sqrt{\bar{\bar{G}}}}{\alpha+\beta\rho_1},\\ \frac{\partial \log\sigma}{\partial v} &= \frac{\alpha+\beta\rho_1}{(\rho_2-\rho_1)(\alpha+\beta\rho_2)}\frac{\partial\rho_2}{\partial v}+\frac{\partial}{\partial v}\log\frac{\beta\rho_2\sqrt{\bar{\bar{E}}}}{\alpha+\beta\rho_2}.\end{aligned}$$

With the aid of equations (24) and (25) we reduce these equations to the form

$$\begin{aligned}\frac{\partial \log\sigma}{\partial u} &= \frac{1}{\beta\rho_1}\frac{\partial\alpha}{\partial u}+\frac{\partial}{\partial u}\log\left(\frac{1}{\rho_2}-\frac{1}{\rho_1}\right)+2\frac{\partial}{\partial u}\log\frac{\beta\rho_1\sqrt{\bar{\bar{G}}}}{\alpha+\beta\rho_1},\\ \frac{\partial \log\sigma}{\partial v} &= \frac{1}{\beta\rho_2}\frac{\partial\alpha}{\partial v}+\frac{\partial}{\partial v}\log\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)+2\frac{\partial}{\partial v}\log\frac{\beta\rho_2\sqrt{\bar{\bar{E}}}}{\alpha+\beta\rho_2}.\end{aligned} \tag{32}$$

In consequence of formulas (20) and (24) and the fact that α is a solution of equation (22) (which also is a consequence of these equations), it follows that

$$\frac{\partial}{\partial v}\left(\frac{1}{\beta\rho_1}\frac{\partial\alpha}{\partial u}\right)=\frac{\partial}{\partial u}\left(\frac{1}{\beta\rho_2}\frac{\partial\alpha}{\partial v}\right).$$

Hence the condition of integrability of equations (32) reduces to (27), and consequently the point M', harmonic to M with respect to the centers of principal curvature of $\bar{S}$, describes a surface upon which the parametric curves form a conjugate system $2, O$. Moreover, from (27) and (30) it follows that this conjugate system has equal point invariants.

Equation (27) may be replaced by

$$\beta+\alpha/\rho_1=\bar{\omega}\sqrt{\bar{\bar{G}}}V, \qquad \beta+\alpha/\rho_2=\bar{\omega}\sqrt{\bar{\bar{E}}}U,$$

where $\bar{\omega}$ is a function to be determined, and U and V are functions of u and v alone respectively. Solving these equations for α and β, we obtain

$$\alpha=\frac{\bar{\omega}(\sqrt{\bar{\bar{G}}}\,V-\sqrt{\bar{\bar{E}}}\,U)\rho_1\rho_2}{\rho_2-\rho_1}, \qquad \beta=\frac{\bar{\omega}(\rho_2\sqrt{\bar{\bar{E}}}\,U-\rho_1\sqrt{\bar{\bar{G}}}\,V)}{\rho_2-\rho_1}.$$

In order that these functions satisfy equations (20), the function ω must be such that

$$\frac{\partial \log \bar{\omega}}{\partial u} + \frac{\sqrt{\bar{\bar{E}}}}{\sqrt{\bar{\bar{G}}}} \frac{U}{V} \frac{\frac{\partial}{\partial u}(1/\rho_1)}{\rho_2 - \rho_1} \rho_1 \rho_2 - \frac{\partial}{\partial u} \log \left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right) = 0,$$

(33)

$$\frac{\partial \log \bar{\omega}}{\partial v} - \frac{\sqrt{\bar{\bar{G}}}}{\sqrt{\bar{\bar{E}}}} \frac{V}{U} \frac{\frac{\partial}{\partial v}(1/\rho_2)}{\rho_2 - \rho_1} \rho_1 \rho_2 - \frac{\partial}{\partial v} \log \left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right) = 0.$$

Expressing the consistency of these equations, we obtain

$$(34) \quad \frac{\partial}{\partial v}\left(\frac{\sqrt{\bar{\bar{E}}}}{\sqrt{\bar{\bar{G}}}} \frac{U}{V} \frac{\frac{\partial}{\partial u}(1/\rho_1)}{\rho_2 - \rho_1} \rho_1 \rho_2 \right) + \frac{\partial}{\partial u}\left(\frac{\sqrt{\bar{\bar{G}}}}{\sqrt{\bar{\bar{E}}}} \frac{V}{U} \frac{\frac{\partial}{\partial v}(1/\rho_2)}{\rho_2 - \rho_1} \rho_1 \rho_2 \right) = 0,$$

which is the characteristic equation of surfaces Ω given by Demoulin* without proof.

3. Properties of surfaces C. Determination of surfaces C_0

From the preceding section we have

Theorem I. *A surface C is characterized by the property that it possesses a conjugate system with equal point invariants and that the point equation admits a solution t such that $x^2 + y^2 + z^2 - t^2$ also is a solution, where $x, y,$ and z are the cartesian coördinates of C.*

We shall investigate these surfaces C, and understand that in what follows the conjugate system $2, O$ with equal point invariants is parametric.

If we put

$$\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2 - \left(\frac{\partial t}{\partial u}\right)^2 = \sigma^2,$$

$$\left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 - \left(\frac{\partial t}{\partial v}\right)^2 = \tau^2,$$

it follows that

$$\frac{\partial}{\partial v} \log \sigma \sqrt{\rho} = 0, \qquad \frac{\partial}{\partial u} \log \tau \sqrt{\rho} = 0.$$

Hence the parameters can be chosen so that

$$\sigma = \tau = 1/\sqrt{\rho}.$$

Accordingly we have

$$(35) \qquad E - \left(\frac{\partial t}{\partial u}\right)^2 = \frac{1}{\rho}, \qquad G - \left(\frac{\partial t}{\partial v}\right)^2 = \frac{1}{\rho}.$$

* Loc. cit., p. 707.

Since by hypothesis $x^2 + y^2 + z^2 - t^2$ is a solution of equation (1), we have also

$$F - \frac{\partial t}{\partial u}\frac{\partial t}{\partial v} = 0. \tag{36}$$

Equations (35) may be replaced by

$$\frac{\partial t}{\partial u} = \sqrt{E - 1/\rho}, \qquad \frac{\partial t}{\partial v} = \sqrt{G - 1/\rho}, \tag{37}$$

if the signs of the parameters u and v are suitably chosen.

Now equation (36) may be written

$$F = (E - 1/\rho)^{\frac{1}{2}}(G - 1/\rho)^{\frac{1}{2}}. \tag{38}$$

Moreover, the consistency of equations (37) requires that

$$\frac{\partial}{\partial v}(E - 1/\rho)^{\frac{1}{2}} = \frac{\partial}{\partial u}(G - 1/\rho)^{\frac{1}{2}}. \tag{39}$$

Equations (38) and (39) characterize a surface C. For, if they are satisfied, the function t defined by (37) satisfies (1) and $x^2 + y^2 + z^2 - t^2$ also is a solution in consequence of (38).

We have shown* that when the point equation of a surface is of the form (1) the following identities hold

$$\begin{aligned} \frac{\partial \sqrt{E}}{\partial v} &= -\sqrt{E}\,\frac{\partial \log \sqrt{\rho}}{\partial v} - \sqrt{G}\cos 2\omega\,\frac{\partial \log \sqrt{\rho}}{\partial u}, \\ \frac{\partial \sqrt{G}}{\partial u} &= -\sqrt{G}\,\frac{\partial \log \sqrt{\rho}}{\partial u} - \sqrt{E}\cos 2\omega\,\frac{\partial \log \sqrt{\rho}}{\partial v}, \end{aligned} \tag{40}$$

where 2ω is the angle between the parametric curves. Also in this case the Christoffel symbols formed with respect to the linear element of the surface represent the following expressions:

$$\begin{aligned} \left\{\begin{matrix}11\\1\end{matrix}\right\} &= \frac{\partial \log \sqrt{E}}{\partial u} + 2\cot 2\omega\,\frac{\partial \omega}{\partial u} - \frac{\sqrt{E}}{\sqrt{G}}\cos 2\omega\,\frac{\partial \log \sqrt{\rho}}{\partial v}, \\ \left\{\begin{matrix}11\\2\end{matrix}\right\} &= \frac{E}{G}\frac{\partial \log \sqrt{\rho}}{\partial v} \quad 2\frac{\sqrt{E}}{\sqrt{G}}\frac{1}{\sin 2\omega}\frac{\partial \omega}{\partial u}, \\ \left\{\begin{matrix}22\\1\end{matrix}\right\} &= \frac{G}{E}\frac{\partial \log \sqrt{\rho}}{\partial u} - 2\frac{\sqrt{G}}{\sqrt{E}}\frac{1}{\sin 2\omega}\frac{\partial \omega}{\partial v}, \\ \left\{\begin{matrix}22\\2\end{matrix}\right\} &= \frac{\partial \log \sqrt{G}}{\partial v} + 2\cot 2\omega\,\frac{\partial \omega}{\partial v} - \frac{\sqrt{G}}{\sqrt{E}}\cos 2\omega\,\frac{\partial \log \sqrt{\rho}}{\partial u}. \end{aligned} \tag{41}$$

* M., p. 419.

We suppose that a surface C is known and we seek a conjugate surface C_0 in the sense of § 2. To this end we introduce three functions a_0, b_0, T_0 by the equations

$$(42)\qquad a_0 = -\frac{T_0}{2\cos\omega}\left(\frac{1}{\sqrt{E}}\frac{\partial t}{\partial u}+\frac{1}{\sqrt{G}}\frac{\partial t}{\partial v}\right),$$

$$b_0 = \frac{T_0}{2\sin\omega}\left(\frac{1}{\sqrt{E}}\frac{\partial t}{\partial u}-\frac{1}{\sqrt{G}}\frac{\partial t}{\partial v}\right).$$

If in the first four of equations (10) we replace the subscript 1 by 0 and require that a_0 and b_0 as given by (42) shall satisfy these equations, the resulting equations of condition are reducible by means of (40) and (41) to

$$\frac{\partial T_0}{\partial u} = m_0(\lambda_0-\rho\theta_0)\frac{\partial t}{\partial u},\qquad \frac{\partial T_0}{\partial v} = m_0(\lambda_0+\rho\theta_0)\frac{\partial t}{\partial v},$$

$$(43)\qquad w_0 D = \frac{m_0}{\rho}(\lambda_0-\rho\theta_0)+T_0\left(\left\{\begin{smallmatrix}11\\1\end{smallmatrix}\right\}\frac{\partial t}{\partial u}+\left\{\begin{smallmatrix}11\\2\end{smallmatrix}\right\}\frac{\partial t}{\partial v}-\frac{\partial^2 t}{\partial u^2}\right),$$

$$w_0 D'' = \frac{m_0}{\rho}(\lambda_0+\rho\theta_0)+T_0\left(\left\{\begin{smallmatrix}22\\1\end{smallmatrix}\right\}\frac{\partial t}{\partial u}+\left\{\begin{smallmatrix}22\\2\end{smallmatrix}\right\}\frac{\partial t}{\partial v}-\frac{\partial^2 t}{\partial v^2}\right).$$

From the definition of the Christoffel symbols* it follows that

$$E\left\{\begin{smallmatrix}11\\1\end{smallmatrix}\right\}+F\left\{\begin{smallmatrix}11\\2\end{smallmatrix}\right\}=\frac{1}{2}\frac{\partial E}{\partial u},\qquad F\left\{\begin{smallmatrix}22\\1\end{smallmatrix}\right\}+G\left\{\begin{smallmatrix}22\\2\end{smallmatrix}\right\}=\frac{1}{2}\frac{\partial G}{\partial v},$$

$$\frac{\partial}{\partial u}\log H\sqrt{\rho}=\left\{\begin{smallmatrix}11\\1\end{smallmatrix}\right\},\qquad \frac{\partial}{\partial v}\log H\sqrt{\rho}=\left\{\begin{smallmatrix}22\\2\end{smallmatrix}\right\},$$

where

$$H=\sqrt{EG-F^2}=\sqrt{EG}\sin 2\omega.$$

In consequence of these identities and equations (35) and (36) we have

$$(44)\qquad \left\{\begin{smallmatrix}11\\1\end{smallmatrix}\right\}\frac{\partial t}{\partial u}+\left\{\begin{smallmatrix}11\\2\end{smallmatrix}\right\}\frac{\partial t}{\partial v}-\frac{\partial^2 t}{\partial u^2}=-\left(\left\{\begin{smallmatrix}11\\1\end{smallmatrix}\right\}+\frac{\partial\log\sqrt{\rho}}{\partial u}\right)\Big/\rho\frac{\partial t}{\partial u}$$
$$=-\frac{\partial}{\partial u}\log H\rho\Big/\rho\frac{\partial t}{\partial u},$$

$$\left\{\begin{smallmatrix}22\\1\end{smallmatrix}\right\}\frac{\partial t}{\partial u}+\left\{\begin{smallmatrix}22\\2\end{smallmatrix}\right\}\frac{\partial t}{\partial v}-\frac{\partial^2 t}{\partial v^2}=-\left(\left\{\begin{smallmatrix}22\\2\end{smallmatrix}\right\}+\frac{\partial\log\sqrt{\rho}}{\partial v}\right)\Big/\rho\frac{\partial t}{\partial v}$$
$$=-\frac{\partial}{\partial v}\log H\rho\Big/\rho\frac{\partial t}{\partial v},$$

and

$$(45)\qquad G\frac{\partial t}{\partial u}-F\frac{\partial t}{\partial v}=\frac{1}{\rho}\frac{\partial t}{\partial u},\qquad F\frac{\partial t}{\partial u}-E\frac{\partial t}{\partial v}=-\frac{1}{\rho}\frac{\partial t}{\partial v}.$$

* E., pp. 152, 153.

If in the last two of equations (10) we replace the subscript 1 by 0 and substitute for w_0 the expression

$$w_0 = T_0/H\rho, \tag{46}$$

the resulting equations are reducible by means of (44) and (45) to

$$\begin{aligned} \frac{DT_0}{H} &= m_0(\lambda_0 - \rho\theta_0) - T_0\Big/\frac{\partial t}{\partial u}\cdot\frac{\partial}{\partial u}\log H\rho, \\ \frac{D''T_0}{H} &= m_0(\lambda_0 + \rho\theta_0) - T_0\Big/\frac{\partial t}{\partial v}\cdot\frac{\partial}{\partial v}\log H\rho, \end{aligned} \tag{47}$$

which in consequence of (44) are equivalent to the last two of equations (43).

If we substitute in the first two of equations (43) the values of $\lambda_0 - \rho\theta_0$ and $\lambda_0 + \rho\theta_0$ given by (47), the resulting equations are reducible by means of (46) to

$$\frac{\partial}{\partial u}\log w_0 = \frac{D}{H}\frac{\partial t}{\partial u}, \qquad \frac{\partial}{\partial v}\log w_0 = \frac{D''}{H}\frac{\partial t}{\partial v}. \tag{48}$$

With the aid of the Codazzi equations* for C the condition of integrability of equations (48) is reducible to

$$\begin{aligned} &\frac{D}{H}\left[\left(\frac{\partial \log\sqrt{\rho}}{\partial v} + \left\{\begin{matrix}22\\2\end{matrix}\right\}\right)\frac{\partial t}{\partial u} + \left(\frac{\partial \log\sqrt{\rho}}{\partial u} - \left\{\begin{matrix}22\\1\end{matrix}\right\}\right)\frac{\partial t}{\partial v}\right] \\ &\qquad = \frac{D''}{H}\left[\left(\frac{\partial \log\sqrt{\rho}}{\partial u} + \left\{\begin{matrix}11\\1\end{matrix}\right\}\right)\frac{\partial t}{\partial v} + \left(\frac{\partial \log\sqrt{\rho}}{\partial v} - \left\{\begin{matrix}11\\2\end{matrix}\right\}\right)\frac{\partial t}{\partial u}\right]. \end{aligned} \tag{49}$$

Because of (35), (36), and (44) the expression in the first parenthesis can be given the form

$$-\rho\frac{\partial t}{\partial v}\left(E\left\{\begin{matrix}22\\1\end{matrix}\right\} + F\left\{\begin{matrix}22\\2\end{matrix}\right\} - \frac{\partial t}{\partial u}\frac{\partial^2 t}{\partial v^2} - \frac{1}{\rho}\frac{\partial \log\sqrt{\rho}}{\partial u}\right).$$

From the general definition of the Christoffel symbols it follows that

$$E\left\{\begin{matrix}22\\1\end{matrix}\right\} + F\left\{\begin{matrix}22\\2\end{matrix}\right\} = \frac{\partial F}{\partial v} - \frac{1}{2}\frac{\partial G}{\partial u}.$$

Hence for the special values (35) and (36) the quantity in the parenthesis vanishes identically. The same is true of the coefficient of D''/H in (49). Recapitulating we see that the function w_0 is given by the quadratures (48), and the other functions a_0; b_0, λ_0, θ_0 follow directly from (42), (46), and (47). Hence we have

THEOREM II. *When a surface C is known, a set of solutions of equations* (1), (2), *and* (10) *can be found by a quadrature; the corresponding transform of C*

* Cf. E., p. 156.

*is a surface C_0 such that lines joining corresponding points on C and C_0 form a normal congruence.**

4. Fundamental functions of surfaces Ω

In this section we determine the expressions for the fundamental coefficients of a surface Ω in terms of the functions for C and the transformation functions by which C_0 is obtained.

If $\bar{S}$ denotes a surface normal to the congruence of joins of corresponding points on C and C_0, its coördinates, $\bar{x}, \bar{y}, \bar{z}$, are expressible thus:

$$\bar{x} = x + t\bar{X}, \qquad \bar{y} = y + t\bar{Y}, \qquad \bar{z} = z + t\bar{Z}, \tag{50}$$

where as follows from (14) and (9)

$$\bar{X} = \frac{x_0 - x}{T_0}\lambda_0 m_0 = \frac{1}{T_0}(a_0 X_1 + b_0 X_2 + w_0 X), \tag{51}$$

and similar expressions for $\bar{Y}$ and $\bar{Z}$.

With the aid of (3) and (43) we obtain from the first of these expressions for $\bar{X}$ the following:

$$\begin{aligned} \frac{\partial \bar{X}}{\partial u} &= -\frac{m_0}{T_0}(\lambda_0 - \rho\theta_0)\left(\frac{\partial x}{\partial u} + \frac{\partial t}{\partial u}\bar{X}\right), \\ \frac{\partial \bar{X}}{\partial v} &= -\frac{m_0}{T_0}(\lambda_0 + \rho\theta_0)\left(\frac{\partial x}{\partial v} + \frac{\partial t}{\partial v}\bar{X}\right). \end{aligned} \tag{52}$$

Making use of these results, we derive from (50)

$$\begin{aligned} \frac{\partial \bar{x}}{\partial u} &= \left[1 - \frac{m_0 t}{T_0}(\lambda_0 - \rho\theta_0)\right]\left(\frac{\partial x}{\partial u} + \frac{\partial t}{\partial u}\bar{X}\right), \\ \frac{\partial \bar{x}}{\partial v} &= \left[1 - \frac{m_0 t}{T_0}(\lambda_0 + \rho\theta_0)\right]\left(\frac{\partial x}{\partial v} + \frac{\partial t}{\partial v}\bar{X}\right). \end{aligned} \tag{53}$$

A comparison of (52) and (53) shows that the principal radii of normal curvature of $\bar{S}$ are given by

$$\rho_1 = \frac{T_0}{m_0(\lambda_0 - \rho\theta_0)} - t, \qquad \rho_2 = \frac{T_0}{m_0(\lambda_0 + \rho\theta_0)} - t.$$

From (35), (36), (42), and (51) we obtain the following identities

$$\Sigma\left(\frac{\partial x}{\partial u} + \frac{\partial t}{\partial u}\bar{X}\right)^2 = \Sigma\left(\frac{\partial x}{\partial v} + \frac{\partial t}{\partial v}\bar{X}\right)^2 = \frac{1}{\rho},$$

$$\Sigma\left(\frac{\partial x}{\partial u} + \frac{\partial t}{\partial u}\bar{X}\right)\left(\frac{\partial x}{\partial v} + \frac{\partial t}{\partial v}\bar{X}\right) = 0.$$

* The limiting case when C_0 is at infinity is considered in § 12.

Hence if $\bar{E}$, $\bar{F}$, $\bar{G}$ and $\bar{D}$, $\bar{D}'$, $\bar{D}''$ denote the first and second fundamental coefficients of $\bar{S}$, we have from (52) and (53)

$$
\begin{aligned}
\sqrt{\bar{E}} &= \left[1 - \frac{m_0 t}{T_0}(\lambda_0 - \rho\theta_0)\right]\frac{1}{\sqrt{\rho}}, \\
\sqrt{\bar{G}} &= \left[1 - \frac{m_0 t}{T_0}(\lambda_0 + \rho\theta_0)\right]\frac{1}{\sqrt{\rho}}, \\
\bar{D} &= \frac{m_0}{T_0}(\lambda_0 - \rho\theta_0)\frac{\sqrt{\bar{E}}}{\sqrt{\rho}}, \qquad D'' = \frac{m_0}{T_0}(\lambda_0 + \rho\theta_0)\frac{\sqrt{\bar{G}}}{\sqrt{\rho}}, \\
\bar{F} &= \bar{D}' = 0.
\end{aligned} \tag{54}
$$

If we make use of equations analogous to (2) and (12), we can obtain without difficulty from the above values for ρ_1 and ρ_2 the expressions

$$
\begin{aligned}
\frac{\partial \rho_1}{\partial u} &= \frac{T_0}{m_0(\lambda_0 - \rho\theta_0)^2}\left(2\rho\frac{\partial \theta_0}{\partial u} + \theta_0\frac{\partial \rho}{\partial u}\right), \\
\frac{\partial \rho_2}{\partial v} &= -\frac{T_0}{m_0(\lambda_0 + \rho\theta_0)^2}\left(2\rho\frac{\partial \theta_0}{\partial v} + \theta_0\frac{\partial \rho}{\partial v}\right).
\end{aligned}
$$

It is readily shown that equation (34) is satisfied by these expressions.

5. Determination of the surfaces C_0'

In § 2 we remarked that $\bar{S}$ is one of the sheets of the envelope of spheres with centers on C and radii equal to t. Let $\bar{S}'$ denote the other sheet. Evidently it also is a surface Ω, on account of the symmetry of the problem. If so, there is a surface C_0' conjugate to C, such that the lines joining corresponding points on C and C_0' are the normals to $\bar{S}'$ (cf. § 2). We seek now the values of the functions a_0', b_0', w_0', θ_0', λ_0', which determine this special transformation.

From equations (15) it follows that

$$
a_0'/T_0' = a_0/T_0, \qquad b_0'/T_0' = b_0/T_0.
$$

From the definition of T_0 and T_0' (cf. equation (12)) it is necessary that

$$
w_0'/w_0 = \pm\, T_0'/T_0.
$$

If we take the upper sign we have

$$
a_0'/a_0 = b_0'/b_0 = w_0'/w_0 = T_0'/T_0 = \sigma,
$$

where σ denotes a factor of proportionality, which is found to be a constant when these values of a_0', b_0', w_0' are substituted in

$$\frac{\partial w_0'}{\partial u} = -\frac{D}{2\sqrt{E}}\left(\frac{a_0'}{\cos\omega} - \frac{b_0'}{\sin\omega}\right),$$

(55)

$$\frac{\partial w_0'}{\partial v} = -\frac{D''}{2\sqrt{G}}\left(\frac{a_0'}{\cos\omega} + \frac{b_0'}{\sin\omega}\right),$$

equations analogous to the last two of (10). From (43) it follows that

$$\lambda_0' = \sigma\lambda_0, \qquad \theta_0' = \sigma\theta_0, \qquad m_0' = m_0,$$

and from equations analogous to (9) we see that in this case C_0 and C_0' coincide.

We consider now the other possibility of sign, and have

$$a_0'/a_0 = b_0'/b_0 = -\,w_0'/w_0 = T_0'/T_0 = \sigma.$$

When these values are substituted in (55), the resulting equations are reducible to

$$\frac{\partial}{\partial u}\log \sigma w_0^2 = 0, \qquad \frac{\partial}{\partial v}\log \sigma w_0^2 = 0.$$

Hence σw_0^2 is a constant, which may be taken equal to unity in all generality and so we have

(56) $$a_0' = a_0/w_0^2, \qquad b_0' = b_0/w_0^2, \qquad w_0' = -\,1/w_0, \qquad T_0' = T_0/w_0^2.$$

From the last two of these equations and (46) we get

(57) $$w_0' = -\,T_0'/H\rho,$$

which is consistent with equations analogous to (48) and those from which it was derived, as there indicated. It is evident that the values for a_0' and b_0' satisfy equations analogous to (42).

When the expression (57) for T_0' is substituted in

$$\frac{\partial T_0'}{\partial u} = m_0'(\lambda_0' - \rho\theta_0')\frac{\partial t}{\partial u}, \qquad \frac{\partial T_0'}{\partial v} = m_0'(\lambda_0' + \rho\theta_0')\frac{\partial t}{\partial v},$$

the resulting equations are reducible by (43), (48), and (56) to

$$m_0'(\lambda_0' - \rho\theta_0') = \frac{m_0}{w_0^2}(\lambda_0 - \rho\theta_0) - \frac{2T_0}{w_0^2}\frac{D}{H},$$

58)

$$m_0'(\lambda_0' + \rho\theta_0') = \frac{m_0}{w_0^2}(\lambda_0 + \rho\theta_0) - \frac{2T_0}{w_0^2}\frac{D''}{H}.$$

From these with the aid of (47) we obtain

$$m_0'(\lambda_0' - \rho\theta_0') = -\frac{m_0}{w_0^2}(\lambda_0 - \rho\theta_0) + 2T_0\Big/w_0^2\frac{\partial t}{\partial u}\cdot\frac{\partial}{\partial u}\log H\rho,$$

(9

$$m_0'(\lambda_0' + \rho\theta_0') = -\frac{m_0}{w_0^2}(\lambda_0 + \rho\theta_0) + 2T_0\Big/w_0^2\frac{\partial t}{\partial v}\cdot\frac{\partial}{\partial v}\log H\rho.$$

It is readily shown that these values satisfy equations for C_0' analogous to the last two of (43).

By making use of (46) we obtain from (58)

$$
\begin{aligned}
m_0' \lambda_0' &= \frac{m_0}{w_0^2}\lambda_0 - \frac{\rho}{w_0}(D'' + D), \\
m_0' \theta_0' &= \frac{m_0}{w_0^2}\theta_0 - \frac{1}{w_0}(D'' - D).
\end{aligned} \tag{60}
$$

With the aid of equations (48) and the Codazzi equations* for C we show that these values for λ_0' and θ_0' satisfy the equations

$$
\frac{\partial \lambda_0'}{\partial u} = -\rho \frac{\partial \theta_0'}{\partial u}, \qquad \frac{\partial \lambda_0'}{\partial v} = \rho \frac{\partial \theta_0'}{\partial v}.
$$

Hence the transformation functions as given by (56), (57), and (60) satisfy all the necessary and sufficient conditions, and we have

THEOREM III. *With a surface C there are associated two unique surfaces C_0 and C_0' such that the lines joining points on either of these surfaces and the corresponding points of C form a congruence normal to a surface Ω.*

The determination of these conjugate surfaces can be effected without quadrature in consequence of the statement at the beginning of the present section, but the determination of the functions λ, θ, a, b, w requires a quadrature.

6. TRANSFORMATIONS K_m OF SURFACES C

We establish now transformations K of a surface C into surfaces of the same kind. Let C_1 be one of the new surfaces and θ_1 the solution of equation (1) determining this transformation. From § 1 it follows that the coördinates x_1, y_1, z_1 are given by equations of the form (3). It is evident that t_1, determined by the equations

$$
\begin{aligned}
\frac{\partial t_1}{\partial u} &= \frac{\rho}{\lambda_1}\left[(t_1 - t)\frac{\partial \theta_1}{\partial u} + \theta_1 \frac{\partial t}{\partial u}\right], \\
\frac{\partial t_1}{\partial v} &= -\frac{\rho}{\lambda_1}\left[(t_1 - t)\frac{\partial \theta_1}{\partial v} + \theta_1 \frac{\partial t}{\partial v}\right],
\end{aligned} \tag{61}
$$

is a solution of the point equation of C_1, namely (4). By the definition of the surface C the function $x^2 + y^2 + z^2 - t^2$ also is a solution of equation (1). If the equations which result when t and t_1 in (61) are replaced by $x^2 + y^2 + z^2 - t^2$ and $x_1^2 + y_1^2 + z_1^2 - t_1^2$, respectively, are satisfied, then C_1 is a surface of the same kind as C. When we express this condition, the resulting equations are reducible to

* E., p. 155.

$$
\begin{aligned}
&W_1\frac{\partial\theta_1}{\partial u}+2\theta_1\left(\sum(x_1-x)\frac{\partial x}{\partial u}-(t_1-t)\frac{\partial t}{\partial u}\right)=0,\\
&W_1\frac{\partial\theta_1}{\partial v}+2\theta_1\left(\sum(x_1-x)\frac{\partial x}{\partial v}-(t_1-t)\frac{\partial t}{\partial v}\right)=0,
\end{aligned}
\tag{62}
$$

where we have put

$$
W_1=\sum(x_1-x)^2-(t_1-t)^2=(T_1/\lambda_1 m_1)^2-(t_1-t)^2, \tag{63}
$$

the latter being a consequence of (9) and (12) and $\sum$ indicating as usual summation with respect to x, y, and z.

By means of (61) and the formulas of § 1 we find

$$
\begin{aligned}
&\frac{\partial W_1}{\partial u}=\frac{\lambda_1+\rho\theta_1}{\lambda_1\theta_1}\frac{\partial\theta_1}{\partial u}W_1=\frac{1}{\lambda_1\theta_1}\left(\lambda_1\frac{\partial\theta_1}{\partial u}-\theta_1\frac{\partial\lambda_1}{\partial u}\right)W_1,\\
&\frac{\partial W_1}{\partial v}=\frac{\lambda_1-\rho\theta_1}{\lambda_1\theta_1}\frac{\partial\theta_1}{\partial v}W_1=\frac{1}{\lambda_1\theta_1}\left(\lambda_1\frac{\partial\theta_1}{\partial v}-\theta_1\frac{\partial\lambda_1}{\partial v}\right)W_1.
\end{aligned}
$$

Hence, to within a constant factor, W_1 is equal to θ_1/λ_1. We consider later the case where this factor is equal to zero. For the case where the constant is not equal to zero we put

$$
W_1=2\theta_1/\lambda_1 m_1. \tag{64}
$$

In consequence of this result and of (7), (8), and (15) equations (62) may be put in the form

$$
\begin{aligned}
&\frac{\partial\theta_1}{\partial u}+\sqrt{E}\Big[(\cos\omega a_1-\sin\omega b_1)+(t_1-t)\frac{m_1\lambda_1}{T_0}(\cos\omega a_0\\
&\qquad\qquad -\sin\omega b_0)\Big]=0,\\
&\frac{\partial\theta_1}{\partial v}+\sqrt{G}\Big[(\cos\omega a_1+\sin\omega b_1)+(t_1-t)\frac{m_1\lambda_1}{T_0}(\cos\omega a_0\\
&\qquad\qquad +\sin\omega b_0)\Big]=0.
\end{aligned}
\tag{65}
$$

Thus it is seen that by the above choice of the constant of integration in (64) the constant m_1 does not now appear with both θ_1 and λ_1, as it does in (9), and consequently it is a significant constant. It is readily found that θ_1, as given by (65), satisfies equation (1), in consequence of (9), (61), and (43).

From (63), (64), and (12) we have

$$
2m_1\theta_1\lambda_1=a_1^2+b_1^2+w_1^2-m_1^2\lambda_1^2(t_1-t)^2. \tag{65'}
$$

Hence we have

THEOREM IV. *When a surface C is known, each set of functions θ_1, λ_1, a_1, b_1, w_1, t_1 satisfying the completely integrable system of equations* (2), (10),

(61), *and* (65), *and in the relation* (65′), *determines a transformation of* C *into a surface of the same kind.*

The general solution of this system of equations involves five essential constants in addition to m_1.

The transformation from C into C_0 (cf. § 2) likewise satisfies these conditions. In fact, in this case equations (62) become

$$\left(\frac{T_0}{\lambda_0 m_0} + t_0 - t\right)\left[\left(\frac{T_0}{\lambda_0 m_0} - t_0 + t\right)\frac{\partial\theta_0}{\partial u} + \frac{2\theta_0\sqrt{E}}{T_0}(\cos\omega\, a_0 - \sin\omega\, b_0)\right] = 0,$$

(66)

$$\left(\frac{T_0}{\lambda_0 m_0} + t_0 - t\right)\left[\left(\frac{T_0}{\lambda_0 m_0} - t_0 + t\right)\frac{\partial\theta_0}{\partial v} + \frac{2\theta_0\sqrt{G}}{T_0}(\cos\omega\, a_0 + \sin\omega\, b_0)\right] = 0.$$

This equation is satisfied if we have

$$t_0 = t - T_0/\lambda_0 m_0, \tag{67}$$

as are also equations analogous to (61).

Since the surfaces C and C_0 bear similar relations to $\bar{S}$, we have analogous to (14) the equation

$$\bar{x} = x_0 + \frac{t_0\lambda_0 m_0}{T_0}(x_0 - x), \tag{68}$$

which is consistent with (14) in consequence of (67).

Conversely we shall show that if the function t_0 of a surface C_0 arising from C by a transformation K_m is in the relation (67) to t for C, then the lines joining corresponding points on C and C_0 form a normal congruence. In fact, if we substitute this value of t_0 in (61) and make use of (13), we get equations (15), which are the necessary and sufficient condition that $\bar{S}$ defined by (14) is normal to the joins of corresponding points on C and C_0.

Comparing (63) and (67), we see that W is zero in this case. Moreover, from the results of the preceding paragraph, it follows that W is zero only in the case of the transformations giving the conjugate surfaces C_0 and C_0'.

7. Generalized theorem of permutability of general transformations K

In our previous paper* we established a theorem of permutability for transformations K. We shall recall the results and make a generalization

* M., p. 406.

which will be of service in the development of the theory of the transformations K for the kind of surfaces under discussion in the present paper. In this section we use the term surface S to denote any surface referred to a conjugate system with equal point invariants.

Suppose that θ_1 and θ_2 are two linearly independent solutions of (1) and that S_1 and S_2 are the surfaces arising from the surface S by the transformations K determined by these functions. The cartesian coördinates x_1, y_1, z_1; x_2, y_2, z_2 of S_1 and S_2 are given by equations of the form (3), namely

$$\begin{aligned} \frac{\partial x_i}{\partial u} &= \frac{\rho}{\lambda_i}\left[(x_i - x)\frac{\partial \theta_i}{\partial u} + \theta_i \frac{\partial x}{\partial u}\right] \\ \frac{\partial x_i}{\partial v} &= -\frac{\rho}{\lambda_i}\left[(x_i - x)\frac{\partial \theta_i}{\partial v} + \theta_i \frac{\partial x}{\partial v}\right] \end{aligned} \qquad (i = 1, 2), \tag{69}$$

where in accordance with (2)

$$\frac{\partial \lambda_i}{\partial u} = -\rho \frac{\partial \theta_i}{\partial u}, \qquad \frac{\partial \lambda_i}{\partial v} = \rho \frac{\partial \theta_i}{\partial v}. \tag{70}$$

From (4) it follows that the functions defined by (69) satisfy the corresponding equation

$$\frac{\partial^2 \phi_i}{\partial u \partial v} + \frac{\partial}{\partial v}\log\frac{\lambda_i}{\sqrt{\rho}\theta_i}\frac{\partial \phi_i}{\partial u} + \frac{\partial}{\partial u}\log\frac{\lambda_i}{\sqrt{\rho}\theta_i}\frac{\partial \phi_i}{\partial v} = 0 \qquad (i = 1, 2). \tag{71}$$

These equations, for $i = 1$ and 2, are satisfied also by the functions θ_{12} and θ_{21} respectively, defined by

$$\begin{aligned} \frac{\partial}{\partial u}(\theta_{ij}\lambda_i) &= \rho\left(\theta_i \frac{\partial \theta_j}{\partial u} - \theta_j \frac{\partial \theta_i}{\partial u}\right) \\ \frac{\partial}{\partial v}(\theta_{ij}\lambda_i) &= -\rho\left(\theta_i \frac{\partial \theta_j}{\partial v} - \theta_j \frac{\partial \theta_i}{\partial v}\right) \end{aligned} \qquad (i, j = 1, 2;\ i \neq j). \tag{72}$$

As thus defined the functions θ_{12} and θ_{21} are determined only to within the additive functions c_i/λ_i respectively, where c_i is an arbitrary constant. Hereafter in speaking of two such functions we assume that the constants are so chosen that

$$\theta_{ij}\lambda_i + \theta_{ji}\lambda_j = 0 \qquad (i \neq j), \tag{73}$$

which evidently is consistent with (72).

If we put

$$\rho_i = \lambda_i^2/\rho\theta_i^2, \qquad (i = 1, 2), \tag{74}$$

equations (71) are of the same form as equation (1). The functions λ_{12} and λ_{21} which together with θ_{12} and θ_{21} give transformations of S_1 and S_2 respectively are given by

$$\frac{\partial \lambda_{ij}}{\partial u} = -\frac{\lambda_i^2}{\rho\theta_i^2}\frac{\partial \theta_{ij}}{\partial u}, \qquad \frac{\partial \lambda_{ij}}{\partial v} = \frac{\lambda_i^2}{\rho\theta_i^2}\frac{\partial \theta_{ij}}{\partial v} \quad (i,j=1,2;\ i\neq j). \tag{75}$$

We have shown* that these equations are satisfied by the functions λ_{ij} given by

$$\lambda_{ij}\,\theta_i = \theta_{ij}\,\lambda_i - \theta_j\,\lambda_i + \theta_i\,\lambda_j \qquad (i\neq j). \tag{76}$$

Moreover, in consequence of (73) we have

$$\lambda_{ij}\,\theta_i + \lambda_{ji}\,\theta_j = 0. \tag{77}$$

When S_1 and S_2 are transformed by means of the fu ctions θ_{12}, λ_{12} and θ_{21}, λ_{21} respectively, the resulting surfaces coincide.† In fact, if this surface be denoted by S_{12}, its cartesian coördinates x_{12}, y_{12}, z_{12} are given by equations of the form

$$\lambda_{12}\,\theta_1\,x_{12} = \theta_{12}\,\lambda_1\,x - \lambda_1\,\theta_2\,x_1 + \lambda_2\,\theta_1\,x\ , \tag{78}$$

or by the equivalent equations

$$\lambda_{21}\,\theta_2\,x_{12} = \theta_{21}\,\lambda_2\,x - \lambda_2\,\theta_1\,x_2 + \lambda_1\,\theta_2\,x_1. \tag{79}$$

We say that the surfaces S, S_1, S_2, S_{12} form a *quatern* in accordance with the theorem of permutability. From (78) it is evident that four corresponding points M, M_1, M_2, M_{12} on these respective surfaces are coplanar.

We extend these results by considering three linearly independent solutions of (1), say θ_1, θ_2, θ_3, and we denote by S_1, S_2, and S_3 the corresponding transforms of S. Furthermore, we denote by S_{12}, S_{13}, and S_{23} the surfaces which form quaterns with the respective groups of surfaces, S, S_1, S_2; S, S_1, S_3; S, S_2, S_3. Since S_{12} and S_{13} are transforms of S_1, we seek the surface S' which forms a quatern with them.

If we denote by θ'_{12}, λ'_{12} and θ'_{13}, λ'_{13} the functions by means of which S_{12} and S_{13} are transformed into S', in accordance with (76) we must have

$$\lambda'_{12}\,\theta_{12} = \theta'_{12}\,\lambda_{12} - \lambda_{12}\,\theta_{13} + \lambda_{13}\,\theta_{12}, \tag{80}$$

and analogously to (78),

$$\lambda'_{12}\,\theta_{12}\,x' = \theta'_{12}\,\lambda_{12}\,x_1 - \lambda_{12}\,\theta_{13}\,x_{12} + \lambda_{13}\,\theta_{12}\,x_{13}. \tag{81}$$

In consequence of (78) and a similar expression for x_{13} equation (81) is equivalent to

$$\lambda'_{12}\,\theta_{12}\,x' = (\theta'_{12}\,\lambda_{12}\,\theta_1 + \theta_{13}\,\lambda_1\,\theta_2 - \theta_{12}\,\lambda_1\,\theta_3)\,\frac{x_1}{\theta_1} - \theta_{13}\,\lambda_2\,x_2 + \theta_{12}\,\lambda_3\,x_3. \tag{82}$$

On the assumption that S' and each of the triples of surfaces S_2, S_{21}, S_{23}

* M., p. 405.

† M., p. 406.

and S_3, S_{31}, S_{32}* form quaterns, the following equations analogous to (82) must hold

$$\lambda'_{21}\,\theta_{21}\,x' = (\theta'_{21}\,\lambda_{21}\,\theta_2 + \theta_{23}\,\lambda_2\,\theta_1 - \theta_{21}\,\lambda_2\,\theta_3)\,\frac{x_2}{\theta_2} - \lambda_1\,\theta_{23}\,x_1 + \lambda_3\,\theta_{21}\,x_3,$$

(83)

$$\lambda'_{31}\,\theta_{31}\,x' = (\theta'_{31}\,\lambda_{31}\,\theta_3 + \theta_{32}\,\lambda_3\,\theta_1 - \theta_{31}\,\lambda_3\,\theta_1)\,\frac{x_3}{\theta_3} - \lambda_1\,\theta_{32}\,x_1 + \lambda_2\,\theta_{31}\,x_2.$$

If equation (82) and the first of (83) be multiplied by λ_1 and λ_2 respectively and the results be added, it follows in consequence of (73) that the coefficient of x_3 vanishes and consequently we have an expression of the form

$$Ax' + Bx_1 + Cx_2 = 0.$$

Since this equation holds also in the y's and z's, the quantities A, B, and C must be zero. Consequently we must have

$$\lambda'_{21} = \lambda'_{12},$$

(84)

$$\theta'_{12}\,\lambda_{12}\,\theta_1 = -\theta_{13}\,\lambda_1\,\theta_2 + \theta_{12}\,\lambda_1\,\theta_3 + \theta_{23}\,\lambda_2\,\theta_1,$$

$$\theta'_{21}\,\lambda_{21}\,\theta_2 = -\theta_{23}\,\lambda_2\,\theta_1 + \theta_{21}\,\lambda_2\,\theta_3 + \theta_{13}\,\lambda_1\,\theta_2.$$

From the definition of the functions θ'_{ij} and λ'_{ij} it follows that

$$\theta'_{ij} = \theta'_{ji}, \qquad \lambda'_{ij} = \lambda'_{ji}.\dagger$$

Hence the first of (84) is true and the second and third are equivalent to one another in consequence of (73) and (77). Furthermore, if we compare the second of (84) with (78), we see that θ'_{12} defined by this equation is a solution of the point equation of S_{12} and from the general theory it necessarily satisfies equations analogous to (72), namely

(85)

$$\frac{\partial}{\partial u}(\theta'_{ij}\,\lambda_{ij}) = \frac{\lambda_i^2}{\rho\theta_i^2}\left(\theta_{ij}\frac{\partial\theta_{ik}}{\partial u} - \theta_{ik}\frac{\partial\theta_{ij}}{\partial v}\right)$$

$$\frac{\partial}{\partial v}(\theta'_{ij}\,\lambda_{ij}) = -\frac{\lambda_i^2}{\rho\theta_i^2}\left(\theta_{ij}\frac{\partial\theta_{ik}}{\partial v} - \theta_{ik}\frac{\partial\theta_{ij}}{\partial v}\right) \qquad (i \neq j \neq k).$$

If we proceed in like manner with (82) and the second of (83), multiplying them respectively by λ_1 and λ_3, we are brought to the equations

$$\lambda_3\,\lambda'_{31}\,\theta_{31} - \lambda_1\,\lambda'_{12}\,\theta_{12} = 0,$$

(86)

$$\theta'_{31}\,\lambda_{31}\,\theta_3 = -\theta_{32}\,\lambda_3\,\theta_1 + \theta_{31}\,\lambda_3\,\theta_1 + \theta_{12}\,\lambda_1\,\theta_3.$$

By means of equations (73) and equations analogous to (80) we find that the first of (86) is a consequence of the second and of (84). Furthermore, as

* From the definition of a surface S_{ij} it follows that S_{ji} is the same surface.

† Cf. the previous footnote.

the second of (86) is analogous to the second of (84), it follows that θ'_{31} so defined satisfies equations (85). Hence all the conditions are satisfied and not only have we shown that S' exists, but the functions determining S' are given without quadrature. Now equations (82) and (83) are reducible to

$$\begin{aligned} x'(\theta_{12}\lambda_3 + \theta_{23}\lambda_2 - \theta_{13}\lambda_2) &= \theta_{23}\lambda_2 x_1 - \theta_{13}\lambda_2 x_2 + \theta_{12}\lambda_3 x_3, \\ x'(\theta_{12}\lambda_1 - \theta_{32}\lambda_1 + \theta_{31}\lambda_2) &= -\theta_{32}\lambda_1 x_1 + \theta_{31}\lambda_2 x_2 + \theta_{12}\lambda_1 x_3, \\ x'(\theta_{13}\lambda_1 - \theta_{23}\lambda_1 + \theta_{21}\lambda_3) &= -\theta_{23}\lambda_1 x_1 + \theta_{13}\lambda_1 x_2 + \theta_{21}\lambda_3 x_3, \end{aligned} \tag{87}$$

which are equivalent to one another.

Hence we have the following generalized theorem of permutability:

THEOREM V. *If S is any surface referred to a conjugate system with equal point invariants, and S_1, S_2, S_3 are three surfaces of the same sort obtained from S by transformations K, and S_{12}, S_{13}, S_{23} are surfaces which together with the respective groups S, S_1, S_2; S, S_1, S_3; S, S_2, S_3 form quaterns, there exists a surface S' such that $(S_1, S_{12}, S_{13}, S')$, $(S_2, S_{12}, S_{23}, S')$, $(S_3, S_{13}, S_{23}, S')$ are quaterns. Moreover, when S_{12}, S_{13}, S_{23} are known, S' can be found without any quadratures.*

8. TRANSFORMATIONS OF SURFACES Ω

We consider now two conjugate surfaces C and C_0. It is our purpose to show that, if C_1 is a transform of C in the sense of § 6, there exists a surface C_{10}, such that the surfaces C, C_1, C_0, C_{10} form a quatern (cf. § 7) and that C_{10} is conjugate to C_1; that is, the lines joining corresponding points on C_1 and C_{10} are normal to a family of surfaces Ω.

From (78) it follows that the coördinates x_{10}, y_{10}, z_{10} of C_{10} are given by equations of the form

$$\lambda_{01}\theta_0 x_{10} = \theta_{01}\lambda_0 x - \lambda_0\theta_1 x_0 + \lambda_1\theta_0 x_1, \tag{88}$$

where θ_{01} and λ_{01} are the functions by which C_0 is transformed into C_{10}. From this it is evident that the function t_{10}, defined by

$$\lambda_{01}\theta_0 t_{10} = \theta_{01}\lambda_0 t - \lambda_0\theta_1 t_0 + \lambda_1\theta_0 t_1, \tag{89}$$

is a solution of the point equation of C_{10}.

We know that the functions

$$\sum x^2 - t^2, \qquad \sum x_1^2 - t_1^2, \qquad \sum x_0^2 - t_0^2,$$

where the summation is with respect to the three coördinates, satisfy the point equations of C, C_1, and C_0 respectively. We wish to show that

$$\lambda_{01}\theta_0\left(\sum x_{10}^2 - t_{10}^2\right) = \theta_{01}\lambda_0\left(\sum x^2 - t^2\right) - \lambda_0\theta_1\left(\sum x_0^2 - t_0^2\right) + \lambda_1\theta_0\left(\sum x_1^2 - t_1^2\right),$$

which is of the form of (88).

If we substitute in the left-hand member of this equation the expressions for x_{10}, y_{10}, z_{10}, t_{10} given by (88) and (89), and make use of the identity (cf. (76))

$$\lambda_{01}\,\theta_0 = \lambda_0\,\theta_{01} - \lambda_0\,\theta_1 + \lambda_1\,\theta_0, \tag{90}$$

we get

$$\begin{aligned}\theta_{01}\{\lambda_0\,\theta_1[\textstyle\sum(x_0 - x)^2 - (t_0 - t)^2] - \lambda_1\,\theta_0[\textstyle\sum(x_1 - x)^2 \\ - (t_1 - t)^2]\} + \lambda_1\,\theta_0\,\theta_1[\textstyle\sum(x_1 - x_0)^2 - (t_1 - t_0)^2] = 0.\end{aligned} \tag{91}$$

From equation (9) and an analogous expression for x_0 we obtain by subtraction

$$x_1 - x_0 = \left(\frac{a_1}{\lambda_1 m_1} - \frac{a_0}{\lambda_0 m_0}\right)X_1 + \left(\frac{b_1}{\lambda_1 m_1} - \frac{b_0}{\lambda_0 m_0}\right)X_2 + \left(\frac{w_1}{\lambda_1 m_1} - \frac{w_0}{\lambda_0 m_0}\right)X.$$

Hence with the aid of (67) we have

$$\begin{aligned}\sum(x_1 - x_0)^2 - (t_1 - t_0)^2 = \frac{T_1^2}{\lambda_1^2 m_1^2} \\ - (t_1 - t)^2 - 2\frac{T_0}{\lambda_0 m_0}(t_1 - t) - 2\frac{\Phi_{10}}{\lambda_0\lambda_1 m_0 m_1},\end{aligned}$$

where we have put

$$\Phi_{10} = a_1 a_0 + b_1 b_0 + w_1 w_0. \tag{92}$$

From the expression for x_0 of the form (9) and from (67) it follows also that

$$\sum(x_0 - x)^2 - (t_0 - t)^2 = 0.$$

With the aid of these results and equations (63) and (64) we can reduce (91) to

$$m_0\lambda_0(\theta_{01} - \theta_1) + T_0 m_1\lambda_1(t_1 - t) + \Phi_{10} = 0. \tag{93}$$

It is necessary that θ_{01} given by (93) shall satisfy equations (72) with $i = 0, j = 1$. In order to establish this result we calculate first the derivatives of Φ_{10}. Making use of equations (10) and similar equations for a_0, b_0, w_0, we find

$$\begin{aligned}\frac{\partial\Phi_{10}}{\partial u} = \sqrt{E}\{m_1(a_0\cos\omega - b_0\sin\omega)(\rho\theta_1 - \lambda_1) \\ + m_0(a_1\cos\omega - b_1\sin\omega)(\rho\theta_0 - \lambda_0)\},\end{aligned} \tag{94}$$

$$\begin{aligned}\frac{\partial\Phi_{10}}{\partial v} = -\sqrt{G}\{m_1(a_0\cos\omega + b_0\sin\omega)(\rho\theta_1 + \lambda_1) \\ + m_0(a_1\cos\omega + b_1\sin\omega)(\rho\theta_0 + \lambda_0)\}.\end{aligned}$$

With the aid of these equations and (64) it is readily found that θ_{01} satisfies the above mentioned conditions identically. Hence this value of θ_{01} serves to determine two surfaces C_1 and C_{10} which with C and C_0 form a quatern.

It remains to be shown that C_{10} is a conjugate surface of C_1, which necessitates the condition

$$t_{10} = t_1 - T_{10}/\lambda_{10}\, m_0, \tag{95}$$

and

$$\frac{\partial T_{10}}{\partial u} = m_0\left(\lambda_{10} - \frac{\lambda_1^2\,\theta_{10}}{\rho\theta_1^2}\right)\frac{\partial t_1}{\partial u}, \qquad \frac{\partial T_{10}}{\partial v} = m_0\left(\lambda_{10} + \frac{\lambda_1^2\,\theta_{10}}{\rho\theta_1^2}\right)\frac{\partial t_1}{\partial v}, \tag{96}$$

from (67) and (43).

From (89), (67), and (95) we have, because of the identity (76) for $i = 0$, $j = 1$,

$$(T_{10} - T_0)\,\theta_1 + (\theta_{01} - \theta_1)\, m_0\,\lambda_0\,(t_1 - t) = 0. \tag{97}$$

With the aid of (90), (76), and (72), we find that T_{10} defined by (97) satisfies (96) identically. Hence all the conditions are satisfied and the lines joining corresponding points are normal to a surface Ω. Consequently we have

THEOREM VI. *If C and C_0 are two conjugate surfaces and C_1 is a surface obtained from C by a transformation K_{m_1}, a surface C_{10} can be found without quadratures which is conjugate to C_1 and in the relation of a transformation K_{m_1} with C_0.*

In accordance with the general theory (§ 7) the lines MM_0 and M_1M_{10} lie in a plane. Let ξ, η, ζ denote the cartesian coördinates of the point P of intersection of these lines. Evidently

$$\xi = x + \sigma\frac{\lambda_0\, m_0}{T_0}(x_0 - x) = x_1 + \frac{\sigma_1\,\lambda_{10}\, m_0}{T_{10}}(x_{10} - x_1), \tag{98}$$

where σ and σ_1 denote the distances from M and M_1 respectively to P.

If x_{10} be replaced by its value from (88), the equality of the two expressions for ξ gives

$$x\left(1 - \frac{\sigma\lambda_0\, m_0}{T_0} + \frac{\sigma_1\,\lambda_0\, m_0\,\theta_{01}}{\theta_1\, T_{10}}\right) + x_0\,\lambda_0\, m_0\left(\frac{\sigma}{T_0} + \frac{\sigma_1}{T_{10}}\right)$$
$$- x_1\left(1 + \sigma_1\,\lambda_0\, m_0\frac{-\theta_{01} + \theta_1}{\theta_1\, T_{10}}\right) = 0.$$

Since similar equations hold in the y's and z's, the expressions in the parentheses must be zero. These equations are equivalent to

$$\sigma = -\frac{T_0\,\theta_1}{m_0\,\lambda_0\,(\theta_{01} - \theta_1)}, \qquad \sigma_1 = -\frac{T_{10}\,\theta_1}{m_0\,\lambda_0\,(\theta_{01} - \theta_1)}. \tag{99}$$

Equations (68) define a surface $\overline{S}$ normal to the congruence determined by C and C_0. A surface $\overline{S}_1$ normal to the congruence determined by C_1 and C_{10} is defined by similar equations. The distances from P to corresponding points $\overline{M}$ and $\overline{M}_1$ on these surfaces are $t - \sigma$ and $t_1 - \sigma_1$ respectively.

In consequence of (97) and (99) we have

$$(100)\qquad t-\sigma = t+\frac{T_0\,\theta_1}{m_0\,\lambda_0(\theta_{01}-\theta_1)} = t_1+\frac{T_{10}\,\theta_1}{m_0\,\lambda_0(\theta_{01}-\theta_1)} = t_1-\sigma_1.$$

Consequently the surfaces $\overline{S}$ and $\overline{S}_1$ are the envelope of a two-parameter family of spheres with lines of curvature in correspondence on the two sheets of the envelope. Since such envelopes of spheres were considered first by Ribaucour,* we say that $\overline{S}_1$ is obtained from $\overline{S}$ by a transformation of the Ribaucour type. We have just seen that $\overline{S}_1$ is found by direct processes after S_1 has been determined. In consequence of Theorem IV we have

THEOREM VII. *Surfaces* Ω *admit transformations of the Ribaucour type into surfaces* Ω, *and the determination of these transformations requires the solution of the completely integrable set of equations* (2), (10), (61), *and* (65).

For the sake of brevity we say that $\overline{S}_1$ is obtained from $\overline{S}$ by a *transformation* A_{m_1}.

When the values of σ and σ_1, given by (99) are substituted in (98), the latter become

$$(101)\qquad \xi = x+\frac{\theta_1}{\theta_{01}-\theta_1}(x-x_0) = \frac{\theta_{01}\,x-\theta_1\,x_0}{\theta_{01}-\theta_1}.$$

In § 5 we remarked that the spheres with centers on C and radii determined by the corresponding values of t are enveloped by $\overline{S}$ and a second surface $\overline{S}'$, which arises from $\overline{S}$ by a transformation of the Ribaucour type. In order that $\overline{S}'$ arise from $\overline{S}$ by a transformation A, it is necessary that equations (101) hold when ξ is replaced by x. Evidently this is impossible.

9. THEOREMS OF PERMUTABILITY FOR TRANSFORMATIONS K_m OF SURFACES C AND FOR TRANSFORMATIONS A_m OF SURFACES Ω

By means of the results of the first part of § 7 we prove

THEOREM VIII. *If surfaces* C_1 *and* C_2 *are obtained from a surface* C *by transformations* K_{m_1} *and* K_{m_2}, *a fourth surface* C_{12} *can be found without quadratures which is in the relation of transformations* K'_{m_2} *and* K'_{m_1} *with* C_1 *and* C_2.

The coördinates of a surface S_{12} forming a quatern with C, C_1, C_2 are given by equations of the form (78), where θ_{21} must satisfy equations (72) with $i=2, j=1$, and the function λ_{21} is given by (76) with $i=2, j=1$. Since t, t_1, and t_2 are solutions of the point equations of C, C_1, and C_2 respectively, the function t_{12}, given by

$$(102)\qquad \lambda_{21}\,\theta_2\,t_{12} = \theta_{21}\,\lambda_2\,t-\lambda_2\,\theta_1\,t_2+\lambda_1\,\theta_2\,t_1,$$

*Journal de mathématiques pures et appliquées, ser. 4, vol. 7 (1891), p. 228.

is a solution of the point equation of S_{12}, as follows from (78). The point equations of C, C_1 and C_2 admit also the respective solutions $\sum x^2 - t^2$, $\sum x_1^2 - t_1^2$, and $\sum x_2^2 - t_2^2$. If the point equation of S_{12} admits the solution $\sum x_{12}^2 - t_{12}^2$, S_{12} is a surface C. This will be the case, if

$$(103)\quad \lambda_{21}\theta_2\left(\sum x_{12}^2 - t_{12}^2\right) = \theta_{21}\lambda_2\left(\sum x^2 - t^2\right) - \lambda_2\theta_1\left(\sum x_2^2 - t_2^2\right) + \lambda_1\theta_2\left(\sum x_1^2 - t_1^2\right).$$

This is reducible by means of (102) to

$$\theta_{21}\{\lambda_2\theta_1[(x_2 - x)^2 - (t_2 - t)^2] - \lambda_1\theta_2[\sum (x_1 - x)^2 - (t_1 - t)^2\} + \lambda_1\theta_1\theta_2[\sum (x_1 - x_2)^2 - (t_1 - t_2)^2] = 0.$$

With the aid of equations of the type (9), (63), (64), and (92) for the surfaces C_1 and C_2 this equation can be put in the form

$$(104)\quad \lambda_2\theta_{21}(m_2 - m_1) = m_2\lambda_2\theta_1 + m_1\lambda_1\theta_2 - \Phi_{12} + m_1 m_2\lambda_1\lambda_2(t_1 - t)(t_2 - t).$$

Making use of equations analogous to (64) and (94), we can show that θ_{21} so defined satisfies equations (72) for $i = 2$, $j = 1$. Consequently S_{12} is a surface C and it is determined without quadratures.

Suppose now that we have a quatern (C, C_1, C_2, C_{12}) and let C_0, C_{10}, C_{20} be surfaces conjugate to C, C_1, C_2 respectively. We propose to show that the surface C' which is the transform of C_{12}, C_{10}, and C_{20} in accordance with Theorem V is conjugate to C_{12}.

From (83) it follows that the coördinates x', y', z' of C' are given by equations of the form

$$x'(\theta_{12}\lambda_0 + \theta_{20}\lambda_2 - \theta_{10}\lambda_2) = \theta_{20}\lambda_2 x_1 - \theta_{10}\lambda_2 x_2 + \theta_{12}\lambda_0 x_0.$$

With the aid of (78) this may be given the form

$$(\theta_{12}\lambda_0 + \theta_{20}\lambda_2 - \theta_{10}\lambda_2)(x' - x_{12}) = (x - x_1)(AB + \lambda_1\theta_2 B - \lambda_2\theta_2 C) + (x_2 - x)\frac{\lambda_2}{\lambda_1}(AC + \lambda_1\theta_1 B - \lambda_2\theta_1 C) + (x - x_0)A(A + \lambda_1\theta_2 - \lambda_2\theta_1),$$

where

$$A = \lambda_1\theta_{12} - \lambda_1\theta_2 + \lambda_2\theta_1, \qquad B = \theta_{20}\lambda_2 + \theta_2\lambda_0, \qquad C = \theta_{10}\lambda_1 + \theta_1\lambda_0.$$

This equation is satisfied also by the y's, z's, and t's.

From equations analogous to (9), (63), (64), (67), and (104) we have

$$\sum (x_i - x)^2 - (t_i - t)^2 = 2\theta_i/\lambda_i m_i \qquad (i = 1, 2),$$

$$\sum (x_0 - x)^2 - (t_0 - t)^2 = 0,$$

$$\sum (x_1 - x)(x_2 - x) - (t_1 - t)(t_2 - t) = \frac{\Phi_{12}}{m_1 m_2 \lambda_1 \lambda_2} - (t_1 - t)(t_2 - t) = \frac{1}{m_1 m_2 \lambda_1 \lambda_2}[A(m_2 - m_1) + m_1\lambda_2\theta_1 + m_2\lambda_1\theta_2],$$

$$\sum (x_1 - x)(x_0 - x) - (t_1 - t)(t_0 - t) = C/m_1 \lambda_1 \lambda_0,$$
$$\sum (x_2 - x)(x_0 - x) - (t_2 - t)(t_0 - t) = B/m_2 \lambda_2 \lambda_0.$$

With the aid of these equations we show that

$$\sum (x' - x_{12})^2 - (t' - t_{12})^2 = 0.$$

Hence (§ 6) C' and C_{12} are conjugate surfaces and from Theorem VI we have

THEOREM IX. *If $\overline{S}_1$ and $\overline{S}_2$ are two surfaces Ω obtained from $\overline{S}$, a given surface Ω, by transformations A_{m_1} and A_{m_2}, there can be found without quadratures a fourth surface Ω which is in the relation of transformations A'_{m_2} and A'_{m_1} with both $\overline{S}_1$ and $\overline{S}_2$.*

We have seen that when a surface C is known, so also are two surfaces Ω, namely the sheets $\overline{S}$ and $\overline{S}'$ of the envelope of spheres of radius t and centers on C. In § 5 we showed that as soon as the surface C_0 has been found, the surface C'_0 follows at once. Since the relations between C and C_0, and C and C'_0 are of the same kind, the theorem of permutability gives a surface C'_{10} such that the surfaces C, C'_0, C_1, C'_{10} are related among themselves in the same manner as C, C_0, C_1, C_{10}. Consequently when a transformation A_{m_1} of a surface $\overline{S}$ into a surface $\overline{S}_1$ is known, there follows a transformation A_{m_1} from $\overline{S}'$ into a surface $\overline{S}'_1$, and the latter transformation can be found without quadratures. Moreover, $\overline{S}_1$ and $\overline{S}'_1$ envelop a two-parameter family of spheres whose centers are on C_1 and whose radii are given by t_1. Hence the four surfaces $\overline{S}$, $\overline{S}'$, $\overline{S}_1$, $\overline{S}'_1$ form a quatern under a theorem of permutability for general transformations of Ribaucour.*

There is a surface $\overline{S}''$ which with $\overline{S}$ constitutes the envelope of the two-parameter family of spheres whose centers are on C_0 and whose radii are given by t_0. In this way we get a quatern $\overline{S}$, $\overline{S}''$, $\overline{S}_1$, $\overline{S}''$ similar to the last one mentioned. In a similar manner other quaterns for general transformations of Ribaucour can be found.

10. PARALLEL SURFACES Ω

In our general discussion of transformations K we pointed out the fact† that when we take $\theta_1 = -\lambda_1 = 1$, the surface S_1 is an associate of S. In this case equations (3) and (61) reduce to

$$\frac{\partial x_1}{\partial u} = -\rho \frac{\partial x}{\partial u}, \qquad \frac{\partial x_1}{\partial v} = \rho \frac{\partial x}{\partial v},$$
$$\frac{\partial t_1}{\partial u} = -\rho \frac{\partial t}{\partial u}, \qquad \frac{\partial t_1}{\partial v} = \rho \frac{\partial t}{\partial v}, \tag{105}$$

* Bianchi was the first to show that there is a theorem of permutability of transformations of Ribaucour.

† L. c., p. 401.

and x_1, y_1, z_1, t_1 are solutions of the equation

$$\frac{\partial^2 \phi}{\partial u \partial v} = \frac{\partial \log \sqrt{\rho}}{\partial v}\frac{\partial \phi}{\partial u} + \frac{\partial \log \sqrt{\rho}}{\partial u}\frac{\partial \phi}{\partial v}. \tag{106}$$

In consequence of (105) and similar equations in y_1 and z_1 we have

$$\frac{\partial}{\partial u}(x_1^2 + y_1^2 + z_1^2 - t_1^2) = -\rho\left(x_1\frac{\partial x}{\partial u} + y_1\frac{\partial y}{\partial u} + z_1\frac{\partial z}{\partial u} - t_1\frac{\partial t}{\partial u}\right),$$

$$\frac{\partial}{\partial v}(x_1^2 + y_1^2 + z_1^2 - t_1^2) = \rho\left(x_1\frac{\partial x}{\partial v} + y_1\frac{\partial y}{\partial v} + z_1\frac{\partial z}{\partial v} - t_1\frac{\partial t}{\partial v}\right).$$

Making use of these results and (36), we can show that $x_1^2 + y_1^2 + z_1^2 - t_1^2$ is a solution of (106). Hence the associate surface of a surface C is also a surface C, and consequently determines a surface Ω. It is our purpose now to discover the relation between the latter surface Ω and the one determined by the given surface C.

If we put $\theta_1 = -\lambda_1 = 1$ in equations (72) with $i = 1, j = 0$, we find with the aid of (70), (73), and (76)

$$\theta_{10} = \lambda_0, \qquad \theta_{01} = 1, \qquad \lambda_{10} = \theta_0, \qquad \lambda_{01} = -1. \tag{107}$$

And the equation analogous to (78) reduces to

$$\lambda_0(x_0 - x) = \theta_0(x_{10} - x_1). \tag{108}$$

From the values for θ_{01} and λ_{01}, it follows that S_0 and S_{10} are associate surfaces. Hence if S_0 is a surface C so also is S_{10}. Suppose now that we have four surfaces C, C_1, C_0, C_{10} forming a quatern such that the pairs C and C_1, C_0 and C_{10} are associate. We wish to show that if C_0 is a conjugate of C so likewise is C_{10} of C_1.

From (108) it follows that the auxiliary function t_{10} of C_{10} is given by

$$\lambda_0(t_0 - t) = \theta_0(t_{10} - t_1). \tag{109}$$

The condition, analogous to (67), that C_{10} be a conjugate of C_1 becomes, in consequence of (107),

$$t_{10} = t_1 - T_{10}/\theta_0 m_0. \tag{110}$$

Hence from (67), (109), and (110) we have

$$T_{10} = T_0. \tag{111}$$

In consequence of (105) and (107) equations (96) are reducible to

$$\frac{\partial T_{10}}{\partial u} = m_0(\lambda_0 - \rho\theta_0)\frac{\partial t}{\partial u}, \qquad \frac{\partial T_{10}}{\partial v} = m_0(\lambda_0 + \rho\theta_0)\frac{\partial t}{\partial v}. \tag{112}$$

When we substitute the value of T_{10} given by (111), the resulting equations

are the same as the first two of (43). Hence all the conditions are satisfied and C_1 and C_{10} are conjugate. Accordingly we have

THEOREM X. *If four surfaces C, C_1, C_0, C_{10} form a quatern such that C and C_1 are associate and likewise C_0 and C_{10}, and if C_0 is a conjugate of C, so also is C_{10} a conjugate of C_1. Moreover, the surfaces Ω normal respectively to the lines joining corresponding points on C and C_0 and on C_1 and C_{10} correspond with parallelism of tangent planes.*

The last part of this theorem follows from (108). Hence when we take $\theta_1 = 1$, the transform of $\bar{S}$ and $\bar{S}$ itself correspond with parallelism of tangent planes instead of constituting the envelope of a two-parameter family of spheres.

From the foregoing results and those of § 9 follows

THEOREM XI. *If $\bar{S}$, $\bar{S}_1$, $\bar{S}_2$, $\bar{S}'$ are four surfaces forming a quatern in accordance with the theorem of permutability of transformations A_m and if $\bar{S}$ and $\bar{S}_1$ correspond with parallelism of tangent planes, so also do $\bar{S}_2$ and $\bar{S}'$.*

11. ISOTHERMIC SURFACES

An isothermic surface is characterized by the property that when its lines of curvature are parametric, the fundamental quantities E and G are equal, or can be made so by a change of the parameters. If we put

$$\sqrt{E} = \sqrt{G} = e^{\omega},$$

equations (24) become

$$(113) \qquad \frac{\partial}{\partial v}\left(\frac{1}{\rho_1}\right) = \left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right)\frac{\partial \omega}{\partial v}, \qquad \frac{\partial}{\partial u}\left(\frac{1}{\rho_2}\right) = \left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)\frac{\partial \omega}{\partial u}.$$

From these results and (34) it follows that an isothermic surface is a surface Ω.

In this case $\bar{S}$ itself is one of the two surfaces C described by points on the normal, since the point equation of $\bar{S}$ has equal invariants and admits the solution $x^2 + y^2 + z^2$. If we call it C, we have $t = 0$. From (42) it follows at once that a_0 and b_0 are zero and from equations analogous to the last two of (10) that w_0 is constant.

Now $\rho = e^{-2\omega}$, and consequently from (43) we have

$$(114) \qquad \theta_0 = \frac{w_0}{2m_0} e^{2\omega}\left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right), \qquad \lambda_0 = \frac{w_0}{2m_0}\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right).$$

If $\bar{S}_1$ also is to be isothermic we must have $t_1 = 0$. Equations (67) and (95) reduce to

$$t_0 = -\, w_0/\lambda_0 m_0, \qquad t_{10} = -\, w_{10}/\lambda_{10} m_0,$$

where evidently w_{10} is a constant. In order that these values may satisfy (89), we must have $w_{10} = w_0$.

For the present case equations (2) are

$$(115) \qquad \frac{\partial \lambda_0}{\partial u} = -e^{-2\omega}\frac{\partial \theta_0}{\partial u}, \qquad \frac{\partial \lambda_0}{\partial v} = e^{-2\omega}\frac{\partial \theta_0}{\partial v}.$$

With the aid of (113) it is readily shown that the functions (114) satisfy equations (115). Darboux* pointed out that this value θ_0 is a solution of the point equation of isothermic surfaces without an indication of its bearing in the present case.

We have applied† the general results of § 1 to the case where S is isothermic and will not repeat any of that investigation. We content ourselves with the observation that the point M_0 harmonic to a point M of an isothermic surface S with respect to the corresponding centers of principal curvature of S generates a surface S_0 upon which the developables of the congruence of normals to S cut out a conjugate system with equal point invariants. Moreover, the transformations D_m of S determine transformations K_m of S_0.

12. Surfaces with isothermal spherical representation of their lines of curvature

There is one class of surfaces Ω whose transformations cannot be handled as in the preceding pages. They are surfaces whose lines of curvature are represented on the gaussian unit sphere by an isothermal system. The normals to such a surface form a congruence of Ribaucour,‡ and the only normal congruence of Ribaucour. The developables of the congruence meet the middle surface of the congruence in a conjugate system with equal point invariants. Hence the middle surface is a surface C and the surface C_0 is at infinity.

Let M be any minimal surface and S a surface corresponding to M with orthogonality of linear elements; for the sake of brevity we say that S is an *ortho-surface* of M. If lines be drawn through points of S parallel to the corresponding normals to M, these lines form a normal congruence of Ribaucour and S is the middle surface. Moreover, this is the most general way of obtaining normal congruences of Ribaucour.§ We consider this case and denote by Σ a surface normal to such a normal congruence of Ribaucour.

Elsewhere‖ we have established transformations of the Ribaucour type of surfaces Σ which are such that if Σ and Σ_1 are two surfaces in the relation of the transformation there considered, and M and M_1 are the minimal surfaces

*Annales de l'école normale supérieure, sér. 3, vol. 16 (1899), p. 503.

† M., pp. 422–428.

‡ E., p. 422.

§ E., pp. 420–422.

‖ These Transactions, vol. 9 (1908), pp. 149–177. Hereafter a reference to this memoir will be of the form M_1, p. —.

associate to Σ and Σ_1 respectively, then M and M_1 are the focal surfaces of a W-congruence. It is our purpose to apply to this case the general results of *M.*, §§ 6, 7, in order to show that the middle surfaces of the congruences of normals to Σ and Σ_1 are surfaces C in the relation of a transformation K_m. In that event the relation between Σ and Σ_1 is essentially that of a transformation A_m.

Consider a minimal surface M referred to its asymptotic lines. Its linear element can be given the form

$$ds'^2 = e^{-2\psi}(du^2 + dv^2), \tag{116}$$

where ψ is a solution of the equation

$$\frac{\partial^2 \psi}{\partial u^2} + \frac{\partial^2 \psi}{\partial v^2} = e^{-2\psi}.^* \tag{117}$$

The direction-cosines, X, Y, Z, of the normal to M are solutions of the equation

$$\frac{\partial^2 w}{\partial u \partial v} + \frac{\partial \psi}{\partial v}\frac{\partial w}{\partial u} + \frac{\partial \psi}{\partial u}\frac{\partial w}{\partial v} = 0. \tag{118}$$

Each solution of this equation determines an ortho-surface of M. In fact, if w_2 is such a solution, here assumed to be linearly independent of X, Y, and Z, the coördinates of the corresponding ortho-surface S are given by the quadratures†

$$\frac{\partial x}{\partial u} = e^{2\psi}\left(w_2\frac{\partial X}{\partial u} - X\frac{\partial w_2}{\partial u}\right), \qquad \frac{\partial x}{\partial v} = -e^{2\psi}\left(w_2\frac{\partial X}{\partial v} - X\frac{\partial w_2}{\partial v}\right). \tag{119}$$

It is readily found that the point equation of S is of the form (1) where now

$$1/\sqrt{\rho} = e^{\psi} w_2. \tag{120}$$

Since w_2 is a solution of (118), there exists a function ϕ_2 defined by

$$\frac{\partial \phi_2}{\partial u} = e^{2\psi}\frac{\partial w_2}{\partial u}, \qquad \frac{\partial \phi_2}{\partial v} = -e^{2\psi}\frac{\partial w_2}{\partial v}. \tag{121}$$

Moreover, ϕ_2 is a solution of (1). And it is easily shown that E, F, G for S have the forms (35) and (36) when we take $t = \phi_2$. Hence S is a surface C.

If X_1, Y_1, Z_1; X_2, Y_2, Z_2 denote the direction-cosines of the tangents to the parametric curves of the spherical representation of M, we have‡

$$\frac{\partial X}{\partial u} = e^{-\psi} X_1, \qquad \frac{\partial X_1}{\partial u} = -\frac{\partial \psi}{\partial v} X_2 - e^{-\psi} X, \qquad \frac{\partial X_2}{\partial u} = \frac{\partial \psi}{\partial v} X_1,$$

$$\frac{\partial X}{\partial v} = -e^{-\psi} X_2, \qquad \frac{\partial X_1}{\partial v} = \frac{\partial \psi}{\partial u} X_2, \qquad \frac{\partial X_2}{\partial v} = -\frac{\partial \psi}{\partial u} X_1 + e^{-\psi} X.$$

* M₁, p. 151.
† M₁, p. 152.
‡ M₁, p. 151.

Hence equations (119) may be written

$$\frac{\partial x}{\partial u} = e^{\psi} w_2 X_1 - \frac{\partial \phi_2}{\partial u} X, \qquad \frac{\partial x}{\partial v} = e^{\psi} w_2 X_2 - \frac{\partial \phi_2}{\partial v} X. \tag{122}$$

From these expressions we find that the direction-cosines of the normal to S, say X_0, Y_0, Z_0, are given by equations of the form

$$X_0 = \frac{e^{\psi} w_2}{H}\left(e^{\psi} w_2 X + \frac{\partial \phi_2}{\partial u} X_1 + \frac{\partial \phi_2}{\partial v} X_2\right),$$

where

$$H^2 = EG - F^2 = e^{2\psi} w_2^2 \left[e^{2\psi} w_2^2 + \left(\frac{\partial \phi_2}{\partial u}\right)^2 + \left(\frac{\partial \phi_2}{\partial v}\right)^2 \right].$$

From (122) with the aid of the above equations we find for the second fundamental coefficients, D, D'', of S the expressions

$$D = \sum X_0 \frac{\partial^2 x}{\partial u^2} = -\frac{e^{2\psi} w_2^2}{H}\left[\frac{\partial^2 \phi_2}{\partial u^2} + w_2 + \frac{\partial \phi_2}{\partial v}\frac{\partial \psi}{\partial v} - \frac{\partial \phi_2}{\partial u}\frac{\partial \psi}{\partial u} \right],$$

$$D'' = \sum X_0 \frac{\partial^2 x}{\partial v^2} = -\frac{e^{2\psi} w_2^2}{H}\left[\frac{\partial^2 \phi}{\partial v^2} - w_2 - \frac{\partial \phi_2}{\partial v}\frac{\partial \psi}{\partial v} + \frac{\partial \phi_2}{\partial u}\frac{\partial \psi}{\partial u} \right].$$

In consequence of these values and (120) equations (47) reduce to

$$T_0 e^{-2\psi} = m_0 w_2 (\rho\theta_0 - \lambda_0), \qquad T_0 e^{-2\psi} = m_0 w_0 (\rho\theta_0 + \lambda_0).$$

Also from the above values and (48) we have

$$w_0 = 1/H\rho w_2, \qquad T_0 = 1/w_2,$$

so that the preceding equations are equivalent to

$$\lambda_0 = 0, \qquad \theta_0 = 1/m_0.$$

Hence, as seen from equations analogous to (9), C_0 is at infinity. Consequently the general formulas hold in this case also.

In accordance with M., § 7 we seek a quatern of W-congruences for which the two focal surfaces M and M_1 are minimal. From the general theory of associate surfaces it follows that each surface associate to a given surface, say S_0, determines a W-congruence for which S_0 is one of the focal surfaces. We have shown* that there exist solutions of equation (118) such that if w_1 is such a solution and ϕ_1 the function given by

$$\frac{\partial \phi_1}{\partial u} = e^{2\psi}\frac{\partial w_1}{\partial u}, \qquad \frac{\partial \phi_1}{\partial v} = -e^{2\psi}\frac{\partial w_1}{\partial v}, \tag{123}$$

these functions satisfy the equations

* M_1, p. 155.

$$\frac{\partial^2 \phi}{\partial u^2} = \frac{\partial \psi}{\partial u}\frac{\partial \phi}{\partial u} - \frac{\partial \psi}{\partial v}\frac{\partial \phi}{\partial v} + me^{2\psi} w + m\phi - w,$$

$$\frac{\partial^2 \phi}{\partial u \partial v} = \frac{\partial \psi}{\partial v}\frac{\partial \phi}{\partial u} + \frac{\partial \psi}{\partial u}\frac{\partial \phi}{\partial v},$$

(124)

$$\frac{\partial^2 \phi}{\partial v^2} = -\frac{\partial \psi}{\partial u}\frac{\partial \phi}{\partial u} + \frac{\partial \psi}{\partial v}\frac{\partial \phi}{\partial v} + me^{2\psi} w - m\phi + w,$$

$$e^{-2\psi}\left[\left(\frac{\partial \phi}{\partial u}\right)^2 + \left(\frac{\partial \phi}{\partial v}\right)^2\right] = 2m\phi w - w^2,$$

where m is a constant; moreover, the associate of M given by w_1 determines a W-congruence for which the second focal surface is minimal. We assume that M_1 is a minimal surface so determined. The linear element of its spherical representation is of the form (116) where now

$$e^{\psi_1} = e^{-\psi}\,\phi_1/w_1.^* \tag{125}$$

Since w_2 is a solution of (118) it determines an associate of M which in turn defines a W-congruence of which M is one focal surface; we call the second focal surface Σ_2, and denote by Σ' the fourth surface such that M, M_1, Σ_2, and Σ' are the focal surfaces of a quatern of W-congruences. In order to find Σ' we note that if we put

$$\sigma = e^{\psi} w \tag{126}$$

equation (118) is replaced by

$$\frac{\partial^2 \sigma}{\partial u \partial v} = e^{-\psi}\frac{\partial^2 e^{\psi}}{\partial u \partial v}\sigma. \tag{127}$$

Hence when w in (126) is replaced by w_1 and w_2 we have the values of σ, namely σ_1 and σ_2, which determine M_1 and Σ_2. The corresponding equations for M_1 and Σ_2 are of the form†

$$\frac{\partial^2 \sigma_i'}{\partial u \partial v} = \sigma_i \frac{\partial^2}{\partial u \partial v}\left(\frac{1}{\sigma_i}\right)\cdot \sigma_i' \qquad (i = 1, 2). \tag{128}$$

Moreover, the respective functions σ_1' and σ_2' which determine Σ' as the second focal sheet of the W-congruences with M_1 and Σ_2 as the first focal sheets respectively are given by M. (92), namely

$$\frac{\partial}{\partial u}(\sigma_i \sigma_i') = -\sigma_i^2 \frac{\partial}{\partial u}\left(\frac{\sigma_j}{\sigma_i}\right), \qquad \frac{\partial}{\partial v}(\sigma_i \sigma_i') = \sigma_i^2 \frac{\partial}{\partial v}\left(\frac{\sigma_j}{\sigma_i}\right) \tag{129}$$

$$(i = 1, 2, j = 1, 2;\ i \neq j).$$

If w_1' denotes the function determining the associate surface of M_1, which

* M_1, p. 157.

† M. (91).

in turn determines the W-congruence whose focal surfaces are M_1 and Σ', then the equation analogous to (126) is

$$\sigma_1' = e^{\psi_1} w_1'. \tag{130}$$

In consequence of (125) we have

$$\sigma_1 \sigma_1' = \phi_1 w_1', \tag{131}$$

so that equations (129) may be replaced by

$$\frac{\partial}{\partial u}(\phi_1 w_1') = -e^{2\psi} w_1^2 \frac{\partial}{\partial u}\left(\frac{w_2}{w_1}\right), \qquad \frac{\partial}{\partial v}(\phi_1 w_1') = e^{2\psi} w_1^2 \frac{\partial}{\partial v}\left(\frac{w_2}{w_1}\right). \tag{132}$$

Just as w_2 determines the ortho-surface S of M, so likewise w_1' determines an ortho-surface S_1 of M_1. In M., § 7, we showed that S and S_1 are in the relation of a transformation K. It is our purpose now to find the equations of this transformation.

If we put

$$\alpha = Xe^{\psi}, \qquad \beta = Ye^{\psi}, \qquad \gamma = Ze^{\psi}, \tag{133}$$

where X, Y, Z are the direction-cosines of the normal to M, it follows from (126) that α, β, γ are solutions of (127).

From M. (84) and the expression for ρ_1 analogous to (120), it follows that the coördinates x_1, y_1, z_1 of S_1 are of the form

$$x_1 = \alpha'/\sqrt{\rho_1} = \alpha' e^{\psi_1} w_1' = \alpha' \sigma_1',$$

where α', β', γ', bear to the direction-cosines of the normal to Σ' a relation analogous to that expressed by (133). If α_1 and α_2 denote similar functions for M_1 and Σ_2 respectively, we have from M. (81) and equations analogous to (133)

$$\alpha_2 = \sqrt{\rho} x = e^{-\psi} x/w_2, \qquad \alpha_1 = X' e^{\psi_1} = X' e^{-\psi} \phi_1/w_1,$$

where X', Y', Z' denote the direction-cosines of M_1.

Between these functions there exists the relation

$$\alpha' = \alpha - \frac{\sigma_2}{\sigma_1}(\alpha_1 - \alpha_2),$$

as follows from M. (96). When the above values are substituted, we obtain

$$x_1 + \sigma_2 e^{\psi_1} X' = x + \sigma_1' e^{\psi} X. \tag{134}$$

Since similar equations hold in the y's and z's, we have that lines through corresponding points of S and S_1 parallel to the corresponding normals to M and M_1 meet in a point. We note that as thus drawn these lines generate normal congruences of Ribaucour.

If r denotes the distance from a point of S to the corresponding point on one of the surfaces orthogonal to the first of these congruences, we must have

$$\sum X \frac{\partial}{\partial u}(x + rX) = 0, \qquad \sum X \frac{\partial}{\partial v}(x + rX) = 0.$$

With the aid of (119) we find that $r = \phi_2 = t$, which shows that, as in the general case, the auxiliary function t measures the distance to the orthogonal surface.

The corresponding function t_1 for S_1 must be ϕ_1' where

$$\frac{\partial \phi_1'}{\partial u} = e^{2\psi_1} \frac{\partial w_1'}{\partial u}, \qquad \frac{\partial \phi_1'}{\partial v} = -e^{2\psi_1} \frac{\partial w_1'}{\partial v}. \tag{135}$$

We have now to show that these values of t and t_1 satisfy equations (61). In the first place we must find the expressions for θ and λ in terms of the functions used in this section.

From M. (85), M. (94), (120), and (131) we have

$$\theta = -\phi_1 w_1'.$$

From (5) and the expression for ρ_1 analogous to (120), namely $\sqrt{\rho_1} = e^{-\psi_1}/w_1'$ we find

$$\lambda = w_1/w_2.$$

Substituting these expressions in (61), we find that both are satisfied provided that

$$\phi_1' = \phi_2 + (w_1' - w_2)\,\phi_1/w_1. \tag{136}$$

It is readily shown that this value of ϕ_1' satisfies equations (135).

In consequence of (126) and (130) equation (136) can be written

$$\phi_1' + \sigma_2 e^{\psi_1} = \phi_2 + \sigma_1' e^{\psi}. \tag{137}$$

From this it follows that the point whose coördinates are of the form (134) is equidistant from the two surfaces normal to the two congruences of Ribaucour just considered, and whose cartesian coördinates are of the form

$$x + \phi_2 X, \qquad x_1 + \phi_1' X', \tag{138}$$

respectively.

The functions ϕ_1 and w_1 satisfying equations (123) and (124) determine the most general minimal surface M_1 such that M and M_1 are the focal surfaces of a W-congruence. From the general theory* we know that w_1 determines an ortho-surface, whose coördinates are given by quadratures similar to (119), and that this surface is associate to a unique ortho-surface of M_1, in which

* E., pp. 417–420.

sense it may be said to determine M_1. In view of these various results we have

THEOREM XIV. *If two minimal surfaces, M and M_1, are the focal surfaces of a W-congruence, and S is any ortho-surface of M other than the one determining M_1, there exists an ortho-surface S_1 of M_1, which can be found without quadratures, such that S and S_1 are two surfaces C in the relation of a transformation K. Moreover, the lines through corresponding points on S and S_1 parallel to the normals to M and M_1 respectively at corresponding points meet in points P and are normal to two surfaces Ω which envelop a two-parameter family of spheres whose centers are the points P.*

Consequently surfaces with isothermal spherical representation of their lines of curvature admit transformations A_m.

PRINCETON UNIVERSITY,
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THE GENERAL THEORY OF CONGRUENCES*

BY

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1. Introduction

The general theory of congruences may be based upon the following considerations. Let the system of partial differential equations

$$(D)\qquad \begin{aligned} y_v &= mz, \qquad z_u = ny, \\ y_{uu} &= a\ y + b\ z + c\ y_u + d\ z_v, \\ z_{vv} &= a'\ y + b'\ z + c'\ y_u + d'\ z_v, \end{aligned}$$

where

$$y_u = \frac{\partial y}{\partial u}, \qquad y_v = \frac{\partial y}{\partial v}, \qquad y_{uu} = \frac{\partial^2 y}{\partial u^2}, \qquad \text{etc.},$$

be completely integrable. It will then have precisely four pairs of linearly independent solutions $(y^{(k)}, z^{(k)})$, $(k = 1, 2, 3, 4)$, such that the general solution will be of the form

$$y = \sum_{k=1}^{4} c^{(k)}\, y^{(k)}, \qquad z = \sum_{k=1}^{4} c^{(k)}\, z^{(k)}.$$

Let $y^{(1)}, \cdots y^{(4)}$ and $z^{(1)}, \cdots z^{(4)}$ be interpreted as the homogeneous coördinates of two points P_y and P_z. As u and v vary, these points will describe two surfaces, S_y and S_z (either or both of which may be degenerate), and the line $P_y P_z$ will generate a congruence whose focal surface consists of the two surfaces S_y and S_z. Moreover the ruled surfaces of the congruence obtained by equating either u or v to a constant will be its developables.†

All congruences whose focal surfaces have two distinct sheets may be studied by this method.

The *invariants* and *covariants* of a system of form (D) are those functions of the coefficients and variables which are left unchanged (absolutely or except for a factor), when system (D) is subjected to any transformation of the form

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† E. J. Wilczynski, *Sur la théorie générale des congruences.* Mémoire couronné par la classe des sciences. Mémoires publiés par la Classe des Sciences de l'Académie Royale de Belgique. Collection en 4°. Deuxième série. Tome III (1911). This paper will hereafter be cited as the Brussels Paper.

$$\bar{y} = \lambda(u)y, \qquad \bar{z} = \mu(v)z, \qquad \bar{u} = \phi(u), \qquad \bar{v} = \psi(v),$$

where λ, μ, ϕ, and ψ are arbitrary functions of the variables indicated. It is easy to see that the invariants and covariants of (D) are intrinsically connected with those properties of the congruence which are invariant under projective transformation. Therefore the theory of congruences based upon a consideration of the invariants and covariants of a system of form (D) is a projective theory.*

It is the purpose of the present paper to enrich the theory of congruences, making use of the analytical basis just indicated, by the introduction of a number of new geometrical concepts. The results, aside from their intrinsic interest, throw a great deal of light upon a number of questions in the theory of surfaces. To mention only one instance: we shall find a geometrical interpretation for the condition which Bianchi expresses by saying that a conjugate system is isothermally conjugate.

2. The axis of a surface point with respect to a given conjugate system. The axis congruence and the axis curves

The fundamental covariants of system (D) are y, z, ρ, and σ, where

$$\rho = y_u - \frac{m_u}{m}y, \qquad \sigma = z_v - \frac{n_v}{n}z. \tag{1}$$

The point P_ρ whose coördinates are given by (1) is on the cuspidal edge of the developable which is formed by the tangents of the curves $v = \text{const.}$ on S_y constructed at the various points of the same curve $u = \text{const.}$ The locus of P_ρ, the surface S_ρ, is therefore the second sheet of the focal surface of the congruence which is composed of the tangents to the curves $v = \text{const.}$ on S_y. Moreover, the curves $u = \text{const.}$ and $v = \text{const.}$ on S_ρ form again a conjugate system, the first Laplacian transform of the given conjugate system on S_y. The point P_σ is related in similar fashion to the congruence composed of the tangents to the curves $u = \text{const.}$ on S_z and to the minus first Laplacian transform.

The four points $P_y P_z P_\rho P_\sigma$ are, in general, not coplanar and we shall make use of them systematically as the vertices of a local tetrahedron of reference, for the purpose of studying the properties of a congruence in the vicinity of one of its lines. To completely define this local coördinate system, we shall say that the coördinates of any point given by an expression of the form

$$x_1 y + x_2 z + x_3 \rho + x_4 \sigma,$$

when referred to the system $P_y P_z P_\rho P_\sigma$ shall be proportional to

$$(x_1, x_2, x_3, x_4).$$

* Brussels Paper, §§ 1–3.

Let us consider the planes which osculate the curves $u = \text{const.}$ and $v = \text{const.}$ on S_y; i. e., the curves along which the developables of the congruence touch the focal sheet S_y. The osculating plane of the curve $v = \text{const.}$ on S_y is determined by the three points whose coördinates are $(y^{(1)}, \cdots y^{(4)})$, $(y_u^{(1)}, \cdots y_u^{(4)})$, and $(y_{uu}^{(1)}, \cdots y_{uu}^{(4)})$, or as we shall henceforth say more briefly, by the three points y, y_u, and y_{uu}. Now we have

$$y_u = \frac{m_u}{m} y + \rho,$$

$$y_{uu} = \left(a + c\frac{m_u}{m}\right) y + \left(b + d\frac{n_v}{n}\right) z + c\rho + d\sigma.$$

Therefore the coördinates of these three points in the local coördinate system $P_y P_z P_\rho P_\sigma$ are

$$(1, 0, 0, 0), \left(\frac{m_u}{m}, 0, 1, 0\right), \quad \text{and} \quad \left(a + c\frac{m_u}{m},\ b + d\frac{n_v}{n},\ c,\ d\right)$$

respectively. Therefore we find

$$dx_2 - \left(b + d\frac{n_v}{n}\right) x_4 = 0 \tag{2}$$

as the equation of their plane. In the same way we find

$$x_3 = 0 \tag{3}$$

as the equation of the plane which osculates the curve $u = \text{const.}$ of S_y at P_y, a result which is geometrically obvious, in so far as this is also the plane tangent to S_z at P_z. Of course the plane (2) is also tangent to S_ρ at P_ρ.

We prefer to write equation (2) in a different form obtained from it by making use of the conditions which the coefficients of (D) must satisfy in order that (D) may be completely integrable. These integrability conditions are as follows:*

$$\text{(I)} \qquad \begin{gathered} c = f_u, \quad d' = f_v, \quad b = -d_v - df_v, \quad a' = -c_u' - c' f_u, \\ W = mn - c'\,d = f_{uv}, \\ m_{uu} + d_{vv} + df_{vv} + d_v f_v - f_u m_u = ma + db', \\ n_{vv} + c_{uu}' + c' f_{uu} + c_u' f_u - f_v n_v = c'\,a + nb', \\ 2m_u n + mn_u = a_v + f_u mn + a'\,d, \\ m_v n + 2mn_v = b_u' + f_v mn + bc', \end{gathered}$$

where f may be any function of u and v.

The third of these integrability conditions enables us to replace (2) by the equivalent equation

* Brussels Paper, p. 17, eq. (12).

$$x_2 + \left(f_v + \frac{d_v}{d} - \frac{n_v}{n}\right) x_4 = 0. \tag{4}$$

The curves u = const. and v = const. form, of course, a conjugate system on S_y. *We shall call the line of intersection of the osculating planes of the two curves of this system which meet at P_y, the axis of the point P_y with respect to this conjugate system.* The equations of this axis are then

$$x_2 + \left(f_v + \frac{d_v}{d} - \frac{n_v}{n}\right) x_4 = 0, \qquad x_3 = 0. \tag{5}$$

Clearly the axis of P_y passes through P_y and all of the axes, formed for all of the points of the surface S_y, form a congruence which we shall call the *axis congruence* of the given conjugate system on S_y.

We now proceed to determine the developables of the axis congruence, or what amounts to the same thing, those curves on S_y which have the property that the axes of all of their points form a developable. These curves are the curves of intersection of S_y and the developables of the axis congruence.

Let us write

$$\tau^{(y)} = -\left(f_v + \frac{d_v}{d} - \frac{n_v}{n}\right) z + \sigma. \tag{6}$$

Then the point whose coördinates are given by $\tau^{(y)}$ is on the axis of P_y. It is, more specifically, the point in which the axis intersects $P_z P_\sigma$, and we may think of the axis as determined by the two points y and $\tau^{(y)}$. The condition that the axis shall generate a developable of the axis congruence may also be expressed in this form; two consecutive axes shall intersect. Now, the axis of the surface point, which belongs to the parameters $u + \delta u$, $v + \delta v$, is obtained by joining the point whose coördinates are given by

$$Y = y + y_u\,\delta u + y_v\,\delta v$$

to the point

$$T^{(y)} = \tau^{(y)} + \tau_u^{(y)}\,\delta u + \tau_v^{(y)}\,\delta v.$$

But we have, making use of (D) and (1),

$$y_u = \frac{m_u}{m} y + \rho, \qquad y_v = mz,$$

$$\tau_u^{(y)} = n\left(\frac{b}{d} + \frac{n_v}{n}\right) y + \left[\left(\frac{b}{d}\right)_u + mn\right] z,$$

$$\tau_v^{(y)} = \left(a' + c'\frac{m_u}{m}\right) y + \left[\left(\frac{b}{d}\right)_v + b' + \frac{n_v}{n}\left(\frac{b}{d} + d'\right)\right] z + c'\rho + \left(\frac{b}{d} + d'\right)\sigma.$$

The coördinates of any point on the line $YT^{(y)}$ are given by an expression of the form $\lambda Y + \mu T^{(y)}$. Therefore, the coördinates of such a point, referred to the local tetrahedron $P_y P_z P_\rho P_\sigma$, will be

$$x_1 = \lambda\left(1 + \frac{m_u}{m}\delta u\right) + \mu\left[n\left(\frac{b}{d} + \frac{n_v}{n}\right)\delta u + \left(a' + c'\frac{m_u}{m}\right)\delta v\right],$$

$$x_2 = \lambda m\delta v + \mu\left[-\left(f_v + \frac{d_v}{d} - \frac{n_r}{n}\right) + \left\{\left(\frac{b}{d}\right)_u + mn\right\}\delta u + \left\{\left(\frac{b}{d}\right)_v + b' + \frac{n_v}{n}\left(\frac{b}{d} + d'\right)\right\}\delta v\right],$$

$$x_3 = \lambda\delta u + \mu c'\,\delta v,$$

$$x_4 = \mu\left[1 + \left(\frac{b}{d} + d'\right)\delta v\right].$$

This point will be on the axis of P_y if and only if its coördinates satisfy equations (5), i. e., if and only if λ, μ, δu, and δv can be chosen so as to satisfy the conditions

$$\lambda m\delta v + \mu\left[\left\{\left(\frac{b}{d}\right)_u + mn\right\}\delta u + \left\{\left(\frac{b}{d}\right)_v + b' - \frac{b}{d}\left(\frac{b}{d} + d'\right)\right\}\delta v\right] = 0,$$

$$\lambda\delta u + \mu c'\,\delta v = 0.$$

But we have, from (I),

$$\frac{b}{d} = -f_v - \frac{d_v}{d}, \qquad d' = f_v,$$

and therefore

$$\left(\frac{b}{d}\right)_u + mn = -f_{uv} - \frac{\partial^2 \log d}{\partial u\partial v} + mn = c'\,d - \frac{\partial^2 \log d}{\partial u\partial v},$$

$$\left(\frac{b}{d}\right)_v + b' - \frac{b}{d}\left(\frac{b}{d} + d'\right) = -\frac{1}{d}(df_{vv} + d_v f_v + d_{vv} - db') = -\frac{m}{d}\left(a + f_u\frac{m_u}{m} - \frac{m_{uu}}{m}\right),$$

the equality of the last two members being established by means of the sixth of the integrability conditions (I). Let us put

$$I = a + f_u\frac{m_u}{m} - \frac{m_{uu}}{m},$$

$$d_1 = d\left(c'\,d - \frac{\partial^2 \log d}{\partial u\partial v}\right), \tag{7}$$

where the notation d_1 has been used to indicate the fact that d_1 is the coefficient which corresponds to d in the system (D_1) obtained from (D) by application of the first Laplace transformation.*

With these notations, the above conditions for intersection of $y\tau^{(y)}$ with $Y\mathrm{T}^{(y)}$ reduce to

* See Brussels Paper, equation (137).

$$\lambda dm\delta v + \mu(d_1\,\delta u - mI\delta v) = 0,$$

(8)

$$\lambda\delta u + \mu c'\,\delta v = 0,$$

giving

(9) $$d_1\,\delta u^2 - mI\delta u\delta v - c'\,dm\delta v^2 = 0$$

as the differential equation of the developables of the axis congruence. If we call the curves in which these developables intersect S_y its *axis curves*, we may also regard (9) as the differential equation of the axis curves on S_y.

If a point P_y moves along one of the axis curves on S_y, the corresponding axis describes a developable and meets the cuspidal edge of this developable in the point $\lambda y + \mu\tau^{(y)}$, where the ratio $\lambda : \mu$ is connected with $\delta u : \delta v$ by means of (8). Let us speak of the two points, determined in this way on the axis, as its *foci*. They are given by the expression $\lambda y + \mu\tau^{(y)}$ where $\lambda : \mu$ is determined by the quadratic equation

$$dm\lambda^2 - mI\lambda\mu - c'\,d_1\,\mu^2 = 0.$$

We may express this more elegantly as follows. *The factors of the quadratic covariant*

(10) $$c'\,d_1\,y^2 - mIy\tau^{(y)} - dm\,(\tau^{(y)})^2$$

determine the foci of the axis which passes through P_y.

If the invariant I is equal to zero, the axis curve tangents and the tangents of the given conjugate system on S_y form a harmonic pencil. Moreover, the foci of the axis in that case divide harmonically the points in which the axis meets the original surface and the line which joins the corresponding points of the first and second Laplace transforms.

The asymptotic lines of S_y are given by the differential equation*

(11) $$d\delta u^2 + m\delta v^2 = 0.$$

Therefore the axis curves can coincide with the asymptotic lines only if the simultaneous conditions

(12) $$I = 0, \qquad \frac{\partial^2 \log d}{\partial u \partial v} - 2c'\,d = 0$$

are satisfied. If the simultaneous invariant of (9) and (11) is equal to zero, the axis curves form a conjugate system. But this condition reduces to

(13) $$\frac{\partial^2 \log d}{\partial u \partial v} = 0,$$

which gives on integration

$$d = U \cdot V,$$

where U is a function of u alone, V a function of v alone. Thus, *the axis curves form a conjugate system on* S_y, *if and only if* d *is a product of a function of* u *alone by a function of* v *alone.*

* Brussels Paper, eq. (111).

If this condition is satisfied, the conjugate system formed by the axis curves will coincide with the original conjugate system of S_y, if and only if the further conditions

$$c'\,d = c'\,m = 0 \tag{14}$$

are fulfilled. If m were to vanish, the whole theory of conjugate systems would be inapplicable since the surface S_y would, in that case, degenerate into a curve. Excluding that case, (14) gives $c' = 0$. Therefore S_z must be developable.* Equations (7) show that d_1 will also vanish on account of (13) and (14), i. e., S_1 is also developable. Thus, if the axis curves of a conjugate net on a non-degenerate surface coincide with the given conjugate net, the first and minus first Laplace transformations both give rise to nets on developable surfaces.

If

$$d_1 = mI = c'\,dm = 0 \tag{15}$$

the axis curves of S_y are indeterminate and the axis congruence reduces to the system of lines through a fixed point. If S_y does not degenerate into a curve, $m \neq 0$, and we must have

$$c'\,d = 0, \qquad d\frac{\partial^2 \log d}{\partial u \partial v} = 0, \qquad I = 0. \tag{16}$$

It remains to state briefly the corresponding formulas for the second sheet S_z of the focal surface. The axis of P_z joins this point to the point whose coördinates are given by

$$\tau^{(z)} = \left(\frac{a'}{c'} + \frac{m_u}{m}\right) y + \rho = -\left(f_u + \frac{c'_u}{c'} - \frac{m_u}{m}\right) y + \rho. \tag{17}$$

The differential equation of the axis curves of the surface S_z is

$$c'\,dn\delta u^2 + nJ\delta u\delta v - c'_{-1}\,\delta v^2 = 0, \tag{18}$$

where

$$J = b' + f_v\frac{n_v}{n} - \frac{n_{vv}}{n}, \qquad c'_{-1} = c'\left(c'\,d - \frac{\partial^2 \log c'}{\partial u \partial v}\right). \tag{19}$$

3. The ray of a surface point with respect to a given conjugate system. The ray congruence and the ray curves

Let us apply the principle of duality to the notions of § 2. The osculating plane of the curve v = const. is replaced by the point of intersection of three consecutive tangent planes of S_y along such a curve v = const. But this is the point P_ρ. Similarly three consecutive tangent planes of S_y along a curve

* Brussels Paper, end of § 4.

u = const. intersect in P_z. Thus, the line $P_\rho P_z$ corresponds dualistically to the axis of the point P_ρ. We shall call the line $P_\rho P_z$, which joins corresponding points of the first and minus first Laplace transformed nets, and which corresponds dualistically to the axis of P_y, *the ray of* P_y. We shall moreover speak of the totality of these rays, as the *ray congruence*, and call the curves on S_y which correspond to the developables of this congruence, *the ray curves of* S_y. Further, we shall speak of the points in which any ray touches the cuspidal edges of the two developables of the ray congruence to which it belongs, as the *foci of the ray*.

By means of a method closely analogous to that of § 2, we find the following results. The foci of any ray are given by an expression of the form $\lambda z + \mu\rho$ where the ratio $\lambda : \mu$ is either root of the quadratic

$$mn\lambda^2 + mI\lambda\mu - dm_1\mu^2 = 0, \tag{20}$$

where

$$m_1 = m\left(mn - \frac{\partial^2 \log m}{\partial u \partial v}\right). \tag{21}$$

This may also be expressed as follows. *The foci of the ray are given by the factors of the quadratic covariant*

$$dm_1 z^2 + mIz\rho - mn\rho^2. \tag{22}$$

The ray curves on S_y *are determined by the differential equation*

$$dmn\delta u^2 - mI\delta u\delta v - m_1\delta v^2 = 0. \tag{23}$$

The simultaneous invariant of (23) and (11) is

$$d\frac{\partial^2 \log m}{\partial u \partial v},$$

and d is different from zero if S_y is not developable. Therefore we obtain the following theorem. *The ray curves form a conjugate net on a non-developable surface, if and only if*

$$\frac{\partial^2 \log m}{\partial u \partial v} = 0. \tag{24}$$

Now, in accordance with the theory of Darboux,* the conjugate system composed of the curves u = const. and v = const. on S_y may be studied by means of the equation

$$y_{uv} = \frac{m_u}{m} y_v + mny, \tag{25}$$

which follows from (D). The Laplace-Darboux invariants of this equation are

$$h = mn, \qquad k = -\frac{\partial^2 \log m}{\partial u \partial v} + mn, \tag{26}$$

* *Théorie des surfaces*, vol. 2.

so that (24) is equivalent to the condition $h = k$. We have found the following new interpretation for this condition.

A conjugate system on a non-developable surface has equal Laplace-Darboux invariants, if and only if its ray curves also form a conjugate system.

The condition $h = k$, or in our notation

$$\frac{\partial^2 \log m}{\partial u \partial v} = 0,$$

has been interpreted, by Darboux, in another quite different fashion. We shall now explain and at the same time generalize Darboux's interpretation.

If we let v remain constant, a line of our congruence will generate a developable as the variable u changes. The point in which this line meets the cuspidal edge of the developable which it describes, will be the point P_z, and the osculating plane of the cuspidal edge at P_z will be the plane of the three points

$$(27) \qquad z, \quad z_u = ny, \quad z_{uu} = \left(n_u + n\frac{m_u}{m}\right) y + n\rho.$$

This plane may be regarded as determined by three consecutive points of the cuspidal edge, and we may determine a two-parameter family of conics each of which passes through these three consecutive points, or in other words, has second order contact with the cuspidal edge at P_z.

Let Z_k be the coördinates of any point of the cuspidal edge in the immediate neighborhood of P_z. Then

$$Z_k = z_k + \frac{\partial z_k}{\partial u}\delta u + \frac{1}{2}\frac{\partial^2 z_k}{\partial u^2}\delta u^2 + \cdots \qquad (k = 1, 2, 3, 4),$$

or, on account of (27),

$$Z_k = \left[n\delta u + \tfrac{1}{2}\left(n_u + n\frac{m_u}{m}\right)\delta u^2 + \cdots \right] y_k + z_k + (\tfrac{1}{2}n\delta u^2)\,\rho_k.$$

Thus, the parametric equations of the cuspidal edge, referred to the tetrahedron $P_y P_z P_\rho P_\sigma$, in the neighborhood of P_z, may be expanded in the form

$$(28) \qquad \begin{aligned} x_1 &= n\delta u + \tfrac{1}{2}\left(n_u + n\frac{m_u}{m}\right)\delta u^2 + \cdots, \\ x_2 &= 1 + \cdots, \qquad x_3 = \tfrac{1}{2}n\delta u^2 + \cdots, \end{aligned}$$

where the omitted terms are of higher than the second order in δu.

In order to find the most general conic of the plane $x_4 = 0$ which has second order contact with the cuspidal edge at P_z, it suffices to determine the coefficients of the quadratic equation

$$A_{11}x_1^2 + \cdots + 2A_{12}x_1x_2 + \cdots = 0$$

in such a way that this equation shall be satisfied by the expressions (28)

up to and including second order terms. We find in this way

$$A_{11}(x_1^2 - 2nx_2x_3) + A_{33}x_3^2 + 2A_{13}x_1x_3 = 0 \tag{29}$$

as the equation of the most general conic which has second order contact with the cuspidal edge of the developable $v = \text{const.}$ at the point P_z.

The point P_ρ stands in the same relation to the curves $u = \text{const.}$ of S_y which connects the point P_z with the curves $v = \text{const.}$ If we give a fixed value to u and allow v to vary, the line $P_y P_\rho$ generates a developable of the first Laplacian transformed congruence whose cuspidal edge is the locus of P_ρ. Since we have

$$\rho_v = \frac{m_1}{m}\rho, \qquad \rho_{vv} = \left(\frac{m_1}{m}\right)_v y + m_1 z,$$

this cuspidal edge is represented, up to and including terms of the second order, by the expansions

$$x_1 = \frac{m_1}{m}\delta v + \tfrac{1}{2}\left(\frac{m_1}{m}\right)_v \delta v^2 + \cdots, \quad x_2 = \tfrac{1}{2}m_1\delta v^2 + \cdots, \quad x_3 = 1 + \cdots, \tag{30}$$

again referred to the coördinate system of $P_y P_z P_\rho P_\sigma$. Equations (30) show that the coefficients of (29) may always be determined so that (29) shall touch the cuspidal edge (30) at P_ρ, but that the contact will be of the second order only if

$$\frac{\partial^2 \log m}{\partial u \partial v} = 0,$$

if we exclude the case $m_1 = 0$ from consideration since, in that case, the surface described by P_ρ would degenerate into a curve.

We have therefore proved Darboux's theorem which states that there exists a conic having second order contact with both of these cuspidal edges if and only if the given conjugate system of curves on S_y has equal invariants. We may combine the two results in the following statement.

Given two one-parameter families of curves ($u = \text{const.}$ and $v = \text{const.}$) forming a conjugate system on a non-degenerate surface S_y. Consider the developables composed of the tangents of curves of one of these families and circumscribed about S_y along a fixed curve of the other family. Let P_z and P_ρ be the points in which the tangents of the curves $u = \text{const.}$ and $v = \text{const.}$, which cross at a point P_y of S_y, meet the cuspidal edges of these circumscribing developables. Moreover, let the loci of P_z and P_ρ for all possible values of u and v be non-degenerate surfaces. If there exists a conic in the tangent plane of S_y at P_y, which has second order contact, at P_z and P_ρ respectively, with the cuspidal edges of both of these circumscribing developables, then the ray curves of S_y with respect to the given conjugate system themselves form a conjugate system and the original conjugate system has equal Laplace-Darboux invariants. Moreover, each of these three properties implies the other two.

In general, i. e., whether $\partial^2 \log m/\partial u \partial v$ is zero or not, we can always find a unique curve of the family (29) which shall have simple contact with the cuspidal edge (30) at P_ρ, viz.;

$$C \equiv x_1^2 - 2nx_2 x_3 = 0. \tag{31}$$

Thus the conic C has second order contact with (28) at P_z and simple contact with (30) at P_ρ. Similarly the conic

$$C' \equiv x_1^2 - 2\frac{m_1}{m^2} x_2 x_3 = 0 \tag{32}$$

has second order contact with (30) at P_ρ and simple contact with (28) at P_z. We shall speak of the two conics in the tangent plane of P_y, which are defined in this way, as the *Darboux conics* of the point P_y, although they have been considered by Darboux only in the special case ($h = k$) in which they coincide. We may then state the previous theorem more briefly by saying that a conjugate system with equal invariants is one for which the Darboux conics of every surface point coincide.

Let us consider any line $x_3 = kx_2$ through the point P_y, and in the tangent plane. Such a line will intersect each of the Darboux conics in a pair of points. If we denote by λ the double ratio of these four points, without breaking up the pairs determined by each of the conics, we shall find

$$\left(\frac{1+\lambda}{1-\lambda}\right)^2 = \frac{\left(\frac{m_1}{m^2 n} + 1\right)^2}{4\frac{m_1}{m^2 n}}, \tag{33}$$

so that λ is independent of k. Clearly λ is an absolute invariant of the conjugate system formed by the curves u = const. and v = const. on S_y. If we prefer a different form of statement, we may say that λ is an absolute invariant of the congruences formed by the tangents of these curves, or of a single one of these congruences since each of them determines the other.

The quantity

$$\delta = \frac{c' d}{mn}$$

is also an absolute invariant of the given conjugate system. If we consider the congruence formed by the tangents of the curves u = const. on S_y, the invariant δ may be defined as follows. To every line of the congruence, there belong two linear complexes (the Wälsch associated complexes of the line); δ is the double ratio of these complexes with respect to the two special linear complexes of their pencil.* We shall speak of δ as the *Wälsch invariant of*

* Brussels Paper, eq. (69).

the curves $u = const.$ The Wälsch invariant of the curves $v = \text{const.}$, is

$$\delta_1 = \frac{c_1' d_1}{m_1 n_1} = \frac{m d_1}{m_1 d}.$$

Let us suppose that the curves $u = \text{const.}$ and $v = \text{const.}$ form a conjugate system on S_y of the particular kind which Bianchi calls an isothermally conjugate system. We can see without much difficulty that this will be the case if and only if

$$\frac{\partial^2 \log d/m}{\partial u \partial v} = 0. \tag{34}$$

In fact, the surface S_y is determined except for projective transformations, by the simultaneous system of partial differential equations

$$\begin{aligned} m y_{uu} - d y_{vv} &= a m y + c m y_u + \left(b - d \frac{m_v}{m} \right) y_v, \\ y_{uv} &= m n y + \frac{m_u}{m} y_v, \end{aligned} \tag{35}$$

obtained from system (D) by differentiation and elimination of z and the partial derivatives of z. But from (35) it is evident that the conjugate system formed by the curves $u = \text{const.}$ and $v = \text{const.}$ will be isothermally conjugate, if and only if (34) is satisfied.

The relative invariant

$$W = mn - c' d$$

vanishes, if and only if the congruence composed of the tangents of the curves $u = \text{const.}$ on S_y is a W congruence.* The invariant

$$W_1 = m_1 n_1 - c_1' d_1$$

has the same significance for the congruence composed of the tangents of the curves $v = \text{const.}$ on S_y. We easily find

$$W_1 - W = \frac{\partial^2 \log d/m}{\partial u \partial v}, \tag{36}$$

a relation which gives rise to a theorem discovered recently by Demoulin and Tzitzéica. *If both of these congruences are W-congruences, the curves $u = const.$ and $v = const.$ form an isothermally conjugate system on S_y and these same properties are possessed by all of the conjugate systems and congruences obtained from the original ones by any number of Laplace transformations.* This result may also be stated as follows. *If the first Laplacian transform of a W-congruence is again a W-congruence, the same is true of all of its Laplacian transforms, and each of these congruences determines, by means of its developables, an isothermally conjugate system on each of its focal surfaces.* I may be per-

* Brussels Paper, eq. (59) and (84).

mitted to mention that I published a similar theorem about congruences belonging to linear complexes in my Brussels paper* several years before the announcement of the above theorem by Demoulin and Tzitzéica. Moreover, although all congruences which belong to linear complexes are W congruences, my theorem is not a special case of Demoulin's. It is of interest however to remark that there exist a number of similar theorems concerning properties of conjugate systems which are preserved by Laplacian transformations.

Let us return to the more general case of an isothermally conjugate system. We now see that the condition (34) for such a system may be written in the form $W_1 = W$, or

$$\frac{m_1}{m^2 n} - \frac{d_1}{dmn} = 1 - \delta,$$

or since $\delta_1 = md_1/m_1 d$, in the form

$$\frac{m_1}{m^2 n} = \frac{1-\delta}{1-\delta_1},$$

or finally, on account of (33), in the form

$$\left(\frac{1+\lambda}{1-\lambda}\right)^2 = \frac{\left(1 - \frac{\delta+\delta_1}{2}\right)^2}{(1-\delta)(1-\delta_1)}. \tag{37}$$

This equation contains an important result. *The condition that a system of curves be isothermally conjugate is equivalent to the simple algebraic relation* (37) *between the absolute invariants* λ, δ, *and* δ_1 *of the given conjugate system where, it should be recalled, each of these invariants has been completely interpreted from the point of view of projective geometry.*

The dual considerations may be carried out as follows. Let $s^{(1)}$, $s^{(2)}$, $s^{(3)}$, $s^{(4)}$ be the co-factors of x_1, x_2, x_3, x_4 in the determinant

$$\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ y^{(1)} & y^{(2)} & y^{(3)} & y^{(4)} \\ y_u^{(1)} & y_u^{(2)} & y_u^{(3)} & y_u^{(4)} \\ y_v^{(1)} & y_v^{(2)} & y_v^{(3)} & y_v^{(4)} \end{vmatrix}$$

Then $s^{(1)}$, $s^{(2)}$, $s^{(3)}$, $s^{(4)}$ are the coördinates of the plane tangent to S_y at P_y, and any one of these four quantities may be represented by a symbol of the form

$$s = D(y, y_u, y_v) = mD(y, y_u, z),$$

where D stands for a determinant of the third order. Similarly

$$r = D(z, z_u, z_v) = nD(z, y, z_v)$$

* Brussels Paper, § 11.

represents the coördinates of the plane tangent to S_z at P_z. If we put

$$Y = \frac{r}{n} e^{-f}, \qquad Z = \frac{s}{m} e^{-f},$$

we find that Y and Z are solutions of the adjoint system of (D), viz.:*

$$(\overline{D}) \qquad \begin{aligned} Y_v &= MZ, \qquad Z_u = NY, \\ Y_{uu} &= A\ Y + B\ Z + C\ Y_u + D\ Z_v, \\ Z_{vv} &= A'\ Y + B'\ Z + C'\ Y_u + D'\ Z_v, \end{aligned}$$

where

$$M = c', \qquad N = d,$$

$$(38) \qquad \begin{aligned} A &= a - c_u, & B &= nd' - n_v, & C &= -c, & D &= n, \\ A' &= mc - m_u, & B' &= b' - d'_v, & C' &= m, & D' &= -d. \end{aligned}$$

The fundamental covariants of $(\overline{D})$ are Y, Z, and

$$(39) \qquad R = Y_u - \frac{M_u}{M} Y, \qquad \Sigma = Z_v - \frac{N_v}{N} Z,$$

and it is easy to see that R and Σ are the coördinates of the planes tangent to S_σ and S_ρ at P_σ and P_ρ respectively.

Any expression of the form

$$(40) \qquad u_1 Y + u_2 Z + u_3 R + u_4 \Sigma$$

will represent a plane. We choose a local system of reference, whose tetrahedron is composed of the four planes Y, Z, R, and Σ, in such a way that, with respect to it, the coördinates of the plane (40) shall be proportional to u_1, u_2, u_3, u_4. It may be noted that the tetrahedron Y, Z, R, Σ does not, in general, coincide with y, z, ρ, σ.

If now we let u remain constant while v changes, the plane Y, which is tangent to S_z at P_z, will generate a one-parameter family of planes which envelop a developable whose cuspidal edge is a curve $u = \text{const.}$ on S_y. Since we have

$$Y_v = MZ, \qquad Y_{vv} = \left(M_v + M \frac{N_v}{N} \right) Z + M\Sigma,$$

the parametric equations of this one-parameter family of planes will be

$$(41) \qquad \begin{aligned} u_1 &= 1 + \cdots, & u_2 &= M\delta v + \tfrac{1}{2} \left(M_v + M \frac{N_v}{N} \right) \delta v^2 + \cdots, \\ u_3 &= 0 + \cdots, & u_4 &= \tfrac{1}{2} M \delta v^2 + \cdots, \end{aligned}$$

where the expansion is complete up to and including terms of the second

* Brussels Paper, § 4.

order. The equation

$$A_{22}(u_2^2 - 2Mu_1u_4) + A_{44}u_4^2 + 2A_{24}u_2u_4 = 0, \tag{42}$$

with arbitrary values for A_{22}, A_{44}, A_{24}, represents the most general quadric cone which has second order contact, along the generator through P_z, with the developable circumscribed about S_z along a curve $u = \text{const.}$

The only cone of this family which also has contact, at P_1, with the developable circumscribed about S_ρ along a curve $v = \text{const.}$, is the cone

$$u_2^2 - 2Mu_1u_4 = 0, \tag{43}$$

whose vertex is the point P_y. In the same way we find the equation of a second cone

$$u_2^2 - 2\frac{N_{-1}}{N^2}u_1u_4 = 0, \tag{44}$$

whose vertex is also at P_y, and which has second order contact at P_ρ with the developable circumscribed about S_ρ along the corresponding curve $v = \text{const.}$ and simple contact with the developable circumscribed about S_z along the corresponding curve $u = \text{const.}$ These cones, whose equations may, on account of (38), be written in the forms

$$u_2^2 - 2c'u_1u_4 = 0, \qquad u_2^2 - 2\frac{d_1}{d^2}u_1u_4 = 0 \tag{45}$$

respectively, shall be called the *Darboux cones* of the point P_y. They coincide if and only if

$$\frac{\partial^2 \log d}{\partial u \partial v} = 0, \tag{46}$$

i. e., if the axis curves form a conjugate net on S_y, or if the Laplace-Darboux invariants of the given conjugate system on S_y are equal when this system is referred to tangential coördinates. It is only this special case which Darboux has considered.

Any line through the point P_y and in the tangent plane, will determine a pair of planes tangent to each of the Darboux cones. Let μ be the cross-ratio of the two pairs of planes thus determined; we find

$$\left(\frac{1+\mu}{1-\mu}\right)^2 = \frac{\left(1 + \frac{d_1}{c'd^2}\right)^2}{4\frac{d_1}{c'd^2}}. \tag{47}$$

In the particular case of an isothermally conjugate system we obtain the relation

$$\left(\frac{1+\mu}{1-\mu}\right)^2 = \frac{\left(\delta\delta_1 - \frac{\delta+\delta_1}{2}\right)^2}{\delta\delta_1(1-\delta)(1-\delta_1)}, \tag{48}$$

an equation which corresponds dualistically to (37).

There is, of course, no difficulty in carrying out corresponding considerations for the surface S_z. We shall confine ourselves to writing down the most important equations.

The differential equation of the ray curves of S_z is found to be

$$-n_{-1}\,\delta u^2 - nJ\delta u\delta v + c'\,mn\delta v^2 = 0, \tag{49}$$

where

$$n_{-1} = n\left(mn - \frac{\partial^2 \log n}{\partial u \partial v}\right), \tag{50}$$

and the foci of the ray $P_y P_\sigma$ of any point P_z of the surface S_z are given by the factors of the covariant

$$c'\,n_{-1}\,y^2 + nJy\sigma - mn\sigma^2. \tag{51}$$

The axis curves and ray curves on S_y coincide if and only if

$$d_1 = dmn, \qquad m_1 = c'\,dm. \tag{52}$$

If S_y is neither degenerate nor developable, these conditions are equivalent to

$$W = mn - c'\,d = \frac{\partial^2 \log m}{\partial u \partial v} = -\frac{\partial^2 \log d}{\partial u \partial v}. \tag{53}$$

4. Some further loci connected with the congruence

It is easy to verify that the axes of P_y and P_z can not intersect. But the osculating planes of the curve $v = \text{const.}$ on S_y and of the corresponding curve $u = \text{const.}$ on S_z intersect in a line which we shall call the *joint axis* of the points P_y and P_z. It is the line which joins the points given by the expressions $\tau^{(y)}$ and $\tau^{(z)}$. The developables of the joint-axis congruence are obtained by integrating the differential equation

$$nd_1\,J\delta u^2 + \left[c'\,dmn\left(f_u + \frac{c'_u}{c'} - \frac{m_u}{m}\right)\left(f_v + \frac{d_v}{d} - \frac{n_v}{n}\right) - c'_{-1}\,d_1 - mnIJ\right]\delta u\delta v + mc'_{-1}\,I\delta v^2 = 0, \tag{54}$$

an equation which may also be regarded as giving the joint-axis curves on the surface S_y.

Similarly, if we define the line joining P_ρ to P_σ as the *joint ray* of P_y, we find

$$(55)\quad mn_{-1}\,I\delta u^2 - \left[c'\,dmn\left(f_u + \frac{c'_u}{c'} - \frac{m_u}{m}\right)\left(f_v + \frac{d_v}{d} - \frac{n_v}{n}\right) - m_1\,n_{-1} - mnIJ\right]\delta u\delta v + m_1\,nJ\delta v^2 = 0$$

as the differential equation of the joint-ray curves on S_y.

The characteristics of the osculating planes of the curves v = const. for constant v and variable u, are of course the tangents of these curves, and the cuspidal edges of the corresponding developables are the curves v = const. themselves.

The one-parameter family composed of the osculating planes of the curves v = const. constructed at all of the points of a fixed curve u = const. on S_y, also fails to give rise to anything essentially new. For the characteristics of this family of planes are the tangents of the curves v = const. on S_ρ and their congruence is therefore one of those obtained from the original congruence by a Laplace transformation.

THE UNIVERSITY OF CHICAGO,
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ON MATRICES WHOSE COEFFICIENTS ARE FUNCTIONS OF A SINGLE VARIABLE

BY

J. H. M. WEDDERBURN

1. The methods usually* employed in reducing to its normal form a matrix whose coefficients are polynomials in a variable λ are of such a nature that it is not at all obvious how they can be extended when polynomials are replaced by analytic functions.† The object of this note is to show that the main theorems on elementary factors are not restricted to matric polynomials but apply without appreciable modification to analytic matric functions.

As vectors are employed freely in the sequel, a short explanation of the notation used is necessary. A *vector*, $x = (\xi_1, \xi_2, \cdots, \xi_n)$, is an ordered set of n coefficients: two vectors are equal if, and only if, their corresponding coefficients are equal. The *sum* of two vectors, $x = (\xi_1, \xi_2, \cdots, \xi_n)$ and $y = (\eta_1, \eta_2, \cdots, \eta_n)$ is defined as

$$x + y = (\xi_1 + \eta_1, \xi_2 + \eta_2, \cdots, \xi_n + \eta_n).$$

The *product* of any number ρ into a vector x is defined as

$$\rho x = (\rho\xi_1, \rho\xi_2, \cdots, \rho\xi_n).$$

Except in this elementary case, multiplication of vectors will not be used. A set of n linearly independent vectors is called a *basis*. Any vector can be expressed linearly in terms of the elements of a basis and, in particular, if e_i is the vector for which $\xi_i = 1$, $\xi_j = 0$ $(j \neq i)$, we have $x = \Sigma_1^n \xi_i e_i$.

If A is a matrix (a_{rs}) and $x = \Sigma\xi_i e_i$, the vector $\Sigma_{i=1}^{i=n}(\Sigma_{j=1}^{j=n} a_{ij}\xi_j)e_i$ is denoted by Ax. We evidently have

$$A(x_1 + x_2) = Ax_1 + Ax_2, \quad (A + B)x = Ax + Bx, \quad A \cdot Bx = (AB)x.$$

A matrix is completely determined when its action on n linearly independent vectors is given. If the determinant of a matrix is not zero, it transforms a set of linearly independent vectors into a linearly independent set; while if its rank is r $(r < n)$, there is a unique complex of vectors every element of

* See for instance Bôcher, *Introduction to Higher Algebra*. p. 262.

† To avoid frequent repetition, a matrix whose coefficients are functions of a variable, λ, will be called a matric function of λ.

which is annihilated by the matrix. The following lemma is an immediate consequence of this.

LEMMA I. *If $x_1, x_2, \cdots, x_n$ is a set of vector functions which are holomorphic and linearly independent for all values of λ lying in a given region R, the determinant of the matrix defined by*

$$x_r = P(\lambda)e_r \qquad (r = 1, 2, \cdots, n)$$

does not vanish for any value of λ in R.

It may be remarked that the matrix $P(\lambda)$ is holomorphic in R and, since $|P(\lambda)|$* does not vanish, $P^{-1}(\lambda)$ is also holomorphic in that region. Also, since

$$e_r = P^{-1}(\lambda)x_r, \qquad (r = 1, 2, \cdots, n),$$

the fixed basis $e_1, e_2, \cdots, e_n$ may be replaced by any set of vector functions which are holomorphic and linearly independent throughout R.

2. In the reduction of a matrix to its normal form, we require the following extension of a well-known algebraic theorem.

LEMMA II. *If $f(\lambda)$ and $g(\lambda)$ are two functions which are holomorphic in R and have no factor† in common in that region, there exist two functions, holomorphic in R, such that*

$$p(\lambda)f(\lambda) + q(\lambda)g(\lambda) \equiv 1.$$

If we expand $1/[f(\lambda)g(\lambda)]$ in terms of its principal parts in a Mittag-Leffler series‡ we get

$$\frac{1}{f(\lambda)g(\lambda)} = F(\lambda) + G(\lambda) + \phi(\lambda), \tag{1}$$

where $F(\lambda)$ and $G(\lambda)$ are the parts of the series arising from the zeros of $f(\lambda)$ and $g(\lambda)$, respectively, which lie in R, and $\phi(\lambda)$ is a function which is holomorphic in R. If we now put

$$p(\lambda) = g(\lambda)[G(\lambda) + \phi(\lambda)], \qquad q(\lambda) = f(\lambda)F(\lambda),$$

both $p(\lambda)$ and $q(\lambda)$ are holomorphic in R and, on multiplying both sides of (1) by $f(\lambda)g(\lambda)$, we have

$$1 = f(\lambda)\cdot g(\lambda)[G(\lambda) + \phi(\lambda)] + g(\lambda)\cdot f(\lambda)F(\lambda)$$
$$= f(\lambda)p(\lambda) + g(\lambda)q(\lambda),$$

as required by the Lemma.

* The determinant of the matrix $P(\lambda)$ is denoted by $|P(\lambda)|$.

† I. e., the two functions have no common zeros in R. Functions which are holomorphic and nowhere zero in R play the same rôle as constants do in the algebraic theory and will therefore not be regarded as factors in this paper.

‡ The series used here is a special case of the "generalized Mittag-Leffler theorem." See Osgood, *Funktionentheorie* vol. 1 (second edition), p. 540, or Mittag-Leffler, Acta Mathematica, vol. 4 (1884), p. 8.

3. LEMMA III. *If $p_1(\lambda), p_2(\lambda), \cdots, p_n(\lambda)$ are n functions which are holomorphic and have no factor common to all in R, and $x_1, x_2, \cdots, x_n$ is a set of vector functions which are holomorphic and linearly independent for all values of λ in R, there exists a matric function $P(\lambda)$, holomorphic in R, for which (i) $|P(\lambda)| \neq 0$ for any λ in R, and (ii)*

$$P(\lambda)x_1 = p_1 x_1 + p_2 x_2 + \cdots + p_n x_n.$$

Assume that the lemma is true for bases of order less than n: then, if l is the H.C.F. of $p_1, p_2, \cdots, p_{n-1}$ and $p_r' = p_r/l$, $(r = 1, 2, \cdots, n-1)$, there exists a matric function, Q_0, relative to the basis $x_1, x_2, \cdots, x_{n-1}$ which is holomorphic and never singular in R and is such that

$$Q_0 x_1 = p_1' x_1 + p_2' x_2 + \cdots + p_{n-1}' x_{n-1}.$$

Let Q_1 be the matrix defined relatively to the basis $x_1, x_2, \cdots, x_n$ by

$$Q_1 x_n = x_n, \qquad Q_1 x_r = Q_0 x_r \quad (r = 1, 2, \cdots, n-1),$$

and set $x_r' = Q_1 x_r$ $(r = 1, 2, \cdots, n)$. Since l and p_n have no common factor in R, we can, as in Lemma II, find two functions, α and β, which are holomorphic in R and are such that* $\alpha l + \beta p_n \equiv 1$. If therefore Q_2 is the matrix defined by

$$Q_2 x_1' = l x_1' + p_n x_n', \qquad Q_2 x_n' = -\beta x_2' + \alpha x_n', \qquad Q_2 x_r' = x_r' \quad (r = 2, \cdots, n-1),$$

we have $|Q_2| = 1$, and $P(\lambda) = Q_2 Q_1$ satisfies the conditions of the lemma since

$$Px_1 = Q_2 Q_1 x_1 = Q_2 x_1' = l x_1' + p_n x_n' = \sum_1^n p_r x_r.$$

Since the lemma is obviously true for $n = 1$, the required result follows immediately by induction.

4. THEOREM. *If $A(\lambda)$ is a matric function of rank r which is holomorphic in a region R, there exist two matric functions, $P(\lambda)$ and $Q(\lambda)$, which are holomorphic and non-singular in R, and are such that*

$$P(\lambda)A(\lambda)Q(\lambda) = \left\| \begin{matrix} E_1(\lambda) & & & & & & \\ & E_2(\lambda) & & & & & \\ & & \cdot & & & & \\ & & & \cdot & & & \\ & & & & E_r(\lambda) & & \\ & & & & & 0 & \\ & & & & & & \cdot \\ & & & & & & & \cdot \\ & & & & & & & & 0 \end{matrix} \right\|,$$

* If $p_n = 0$, then $l = 1 = \alpha$.

where $E_1(\lambda), \cdots E_r(\lambda)$ *are functions of* λ *which are holomorphic in* R *and are such that* E_s *is a factor of* E_t *when* $s < t\ (s, t = 1, 2, \cdots, r)$.

This theorem is obviously true when $n = 1$ so that we may make its proof depend on induction. We assume therefore that it is true for matrices of order less than n.

If the rank of A is less than n, there is at least one vector function x, holomorphic in R, for which $Ax \equiv 0$ in R; and we can insure that x itself does not vanish by removing any common factor from its coefficients when expressed in terms of $e_1, e_2, \cdots, e_n$. By Lemma III, there is a matric function, Q_0, holomorphic and non-singular in R, for which $x = Q_0 e_n$, whence $AQ_0 e_n \equiv 0$: the elements in the last column of AQ_0 are therefore all zero. Considering now the conjugate matrix $Q_0' A'$, we see in the same manner that there is a matric function P_0' holomorphic and non-singular in R, which is such that the coefficients in the last column of $Q_0' A'$ are also all zero. It follows then that $P_0 AQ_0$ has zeros both in the last column and in the last row, and may therefore be regarded as a matrix of order $n - 1$ relative to the basis $e_1, e_2, \cdots, e_{n-1}$. There are therefore by hypothesis two matric functions, P_1 and Q_1, of order $n - 1$ which are holomorphic and non-singular in R and are such that $P_1 P_0 AQ_0 Q_1$ has the desired normal form with regard to the basis $e_1, e_2, \cdots, e_{n-1}$: and we have only to extend P_1 and Q_1 to the original basis, $e_1, e_2, \cdots, e_n$, by adding the conditions $P_1 e_n = e_n = Q_1 e_n$ in order to have A in the required normal form with regard to this basis.

Assume now that the rank of A is n. If the coefficients of A have a H.C.F., $f(\lambda)$, we may write $A(\lambda) = f(\lambda) A_1(\lambda)$: therefore, since multiplying $P(\lambda) A(\lambda) Q(\lambda)$ by a scalar factor still leaves it in the normal form, we may, without loss of generality, assume that there is no common factor.

If $x = (\xi_1, \xi_2, \cdots, \xi_n)$ is a constant vector, Ax can only vanish for values of λ for which $|A| = 0$: but the values of λ for which $Ax = 0$ are necessarily continuous functions of the ξ's and, since the coefficients of A have no common factor in R, there are therefore no values of λ in R for which $Ax = 0$ for every constant vector x; hence there is some constant vector, x, for which $Ax \neq 0$ for any value of λ in R. Let X be a constant matrix whose first column consists of the coördinates of this vector, the remaining coefficients being so chosen as to make the determinant of X not zero. The coefficients in the first column of AX are then the coefficients of Ax and therefore have no factor in common.

Let $AXe_1 = \Sigma a_{r1}(\lambda) e_r$. By Lemma III, we can find a matric function $P_1^{-1}(\lambda)$, holomorphic and non-singular in R, which is such that

$$P_1^{-1} e_1 = \sum a_{r1}(\lambda) e_r,$$

whence $P_1 AXe_1 = e_1$. If, therefore,

$$P_1 AXe_r = c_{r1}\, e_1 + \sum_{s=2}^{n} c_{rs}\, e_s \qquad (r = 2, 3, \cdots, n),$$

and Q_1 is the matrix defined by

$$e_1 = Q_1\, e_1, \qquad e_r - c_{r1}\, e_1 = Q_1\, e_r \qquad (r = 2, 3, \cdots, n),$$

then Q_1 is a matric function, holomorphic in R, for which $|Q_1| = 1$ and

$$P_1 AXQ_1\, e_1 = e_1, \qquad P_1 AXQ_1\, e_r = P_1 AX\,(e_r - c_{r1}\, e_1) = \sum_{s=2}^{n} c_{rs}\, e_s$$
$$(r = 2, 3, \cdots, n).$$

All the coefficients in the first row and column of $P_1 AXQ_1$ are therefore zero, except the first, which is 1. Striking out this row and column, we have a matrix of order $n - 1$, and, as we have assumed the theorem true for matrices of order less than n, this matrix can be reduced to the required form by means of two matric functions P_2 and Q_2 which are defined relative to the basis $e_2, e_3, \cdots, e_n$ and are holomorphic and non-singular in R. If we extend these matrices to the basis $e_1, e_2, \cdots, e_n$ as above by adding the conditions $P_2\, e_1 = e_1 = Q_2\, e_1$, the matrix $P_2 P_1 AXQ_1 Q_2$ has the form required by the theorem since in multiplying by P_2 and Q_2 the first row and column of $P_1 AXQ_1$ remain unaltered; the induction is therefore complete.

CONFORMAL CLASSIFICATION OF ANALYTIC ARCS OR ELEMENTS: POINCARÉ'S LOCAL PROBLEM OF CONFORMAL GEOMETRY*

BY

EDWARD KASNER

§ 1. Statement of the Problem

In the geometry based on the infinite group of conformal transformations of the plane (or on the equivalent theory of analytic functions of one complex variable), two types of problems must be carefully distinguished: those relating to *regions* and those relating to *curves* or arcs.

Two *regions* of the plane are equivalent when there exists a conformal representation of the one on the other, the representation to be regular at every interior point. The classic Riemann theory shows that all simply connected regions are equivalent, any one being convertible into say the unit circle. The difficulties connected with the behavior of the boundary (which may be a Jordan curve or a more general point set) have been cleared up in the recent papers of Osgood, Study, and Caratheodory.

Logically simpler problems relating to *curves* or *arcs* have received very scant attention. Two arcs are equivalent provided the one can be converted into the other by a conformal transformation, the transformation to be regular at the points of the arcs, and therefore in some (unspecified) regions including the arcs in their interiors.

The main problem hitherto discussed by the writer in his papers on conformal geometry is the invariant theory of *curvilinear angles.*† Such a configuration (which may be designated also as an analytic angle) consists of two arcs through a common point, both arcs being real, analytic, and regular at the vertex.‡ In this theory it is necessary to distinguish rational and irrational angles. If θ denotes the magnitude of the angle (invariant of first order), then when θ/π is rational there exists a unique conformal invariant

* Presented to the Society, October 25, 1913.

† See *Conformal geometry*, Proceedings of the fifth international congress, Cambridge (1912), vol. 2, pp. 81–87. See also G. A. Pfeiffer's Columbia dissertation, to be published in the American Journal of Mathematics, October, 1915.

‡ The author has also carried out the theory for analytic angles in the complex domain, the sides being regular arcs, real or imaginary. The new feature which then arises is that certain imaginary angles have an *infinite* number of conformal invariants.

of higher order, involving the curvatures and a certain number of higher derivatives of the two curved sides of the angle. On the other hand, if θ/π is irrational, no such higher invariant exists. The transformation is of course assumed to be regular in the neighborhood of the vertex.

The object of the present paper is to study an even simpler and more fundamental problem: the equivalence theory of a *single curve* or arc. When can one analytic arc be converted into another analytic arc by a conformal transformation of the plane? It is apparently implied, in the current literature, that there is no problem here; for any curve (it is implied) can be converted into any other—in particular, into the axis of reals. But this is based on the assumption (not usually stated) that the arcs are real and regular. If we give up either or both of these assumptions we have actual problems which certainly seem worthy of treatment. Our subject (roughly) is *the invariant theory of a single general analytic arc.*

More exactly, the configuration we shall discuss is not an analytic arc but rather that arc together with a specific point of the arc. This (compound) configuration we shall term an *analytic element.* It consists of a point (called base point, which we shall throughout this paper take as origin) and an analytic arc through the point. It may be described also as a *differential element of infinite order.**

Our problem is then precisely what Poincaré has called the LOCAL PROBLEM† of conformal geometry: Given in the first plane (the plane of $z = x + iy$) a point o and an analytic arc l passing through o, and in the second plane (the plane of $Z = X + iY$) a point O and an analytic arc L passing through O; is it possible to find a conformal transformation, that is, is it possible to find Z as an analytic function of z, so as to convert o into O and l in L, the function to be regular in the neighborhood of $z = 0$? This means that we are to find the integral power series

$$Z = c_1 z + c_2 z^2 + c_3 z^3 + \cdots$$

with its first coefficient c_1 different from zero.

Poincaré dismisses this local problem with the remark that there exist

* The writer has introduced elsewhere the concept of *divergent* differential element of infinite order: this corresponds to a divergent power series and may be represented by a non-analytic arc having specified values for all the successive derivatives. Thus to every power series corresponds a geometric entity which may be real or imaginary, regular or irregular, convergent or divergent. This entity is the most general differential element. If it is convergent we call it an analytic element, or, more loosely, an analytic arc or curve.

† As distinguished from the Riemann problem which Poincaré calls the *problème étendue.* See Palermo Rendiconti, vol. 22 (1907), pp. 185–220. Poincaré's object is here to extend both problems to the theory of analytic functions of two complex variables (four-dimensional space).

always an infinitude of solutions.* Obviously he was assuming not merely the *reality* of the arcs considered, which is of course natural since conformal geometry ordinarily means the geometry of the real Gauss plane,—but also the *regularity* of the arcs in the neighborhood of the given points, a very restrictive assumption.

The most general analytic element, if we take the given point o as origin, is represented by writing x and y as integral power series in a parameter t, without absolute terms (that is terms of degree zero). If we eliminate t we obtain y as a series in x which may proceed according to integral or fractional powers of x. If the coefficients are real the element is called *real*; otherwise it is called *imaginary*. If fractional exponents enter and can not be avoided by interchanging x and y (this will then necessarily be the case for *any* choice of rectangular axes), we call the element *irregular*; otherwise the element is *regular*.†

What Poincaré had in mind was the familiar fact that all real regular elements are equivalent: any such element can be reduced conformally to the canonical form $y = 0$ (that is the axis of reals, together, of course, with the origin as base point), and this in an infinitude of ways.

Our new problem is to classify, with respect to the general conformal group, all analytic elements, real and imaginary, regular and irregular.

That distinctions arise in the imaginary domain is obvious, since minimal lines cannot be converted into other lines. For imaginary regular elements the problem is very simple, since it is necessary simply to consider the order of contact of the given element (o, l) with the minimal lines through the given point. It may be discussed synthetically, though for uniformity of treatment we give below (§ 7) the analytic discussion.

But for irregular elements, even in the real plane, the results we find are fairly complicated. It is clear, for example, that the cuspidal element $y = x^{\frac{3}{2}}$ cannot be converted into the regular element $y = 0$, nor into the irregular element $y = x^{\frac{4}{3}}$, for these curves differ qualitatively in an obvious way (in the nature of the singular point at the origin). But suppose the two proposed

* Poincaré shows that the analogous problem in four-dimensional space (in connection with functions of two variables) has in general no solution, but may in special cases have either a finite or an infinite number of solutions.

† See the systematic definitions in Study's *Vorlesungen über Geometrie*, Heft 1, §§ 5, 12. Study however is dealing with analytic curves, not analytic elements; so he speaks of the regular and irregular points (Stellen) of the curve, while we apply the adjectives to the elements (or sometimes to the arcs or curves belonging to the elements). There is no actual ambiguity however. An ordinary node, it should be noticed, is *not* an example of an irregular element, but comes rather under the concept of an analytic angle: we have in fact merely two regular arcs with a common point, that is, two distinct regular elements. An ordinary cusp is the most familiar instance of an irregular element. Any algebraic singularity may be resolved into a number of regular and irregular elements.

elements have the same kind of irregularity (in a sense to be later defined, depending on agreement of certain exponents, certain arithmetic invariants), will they necessarily be equivalent? If not, certain combinations of the coefficients will be invariant, that is, there will be absolute or differential invariants. For example, it turns out that every differential element of the form

$$y = x^{\frac{3}{2}} + \gamma_4 x^{\frac{4}{2}} + \gamma_5 x^{\frac{5}{2}} + \cdots$$

can be (formally) reduced to $y = x^{\frac{3}{2}}$; on the other hand, not every element of form

$$y = x^{\frac{4}{2}} + \gamma_5 x^{\frac{5}{2}} + \gamma_6 x^{\frac{6}{2}} + \cdots$$

can be reduced to $y = x^{\frac{4}{2}}$. Hence in the first type there are no invariants; in the second type there exist invariants—in fact an infinitude of them.

In general, irregular types of elements have absolute invariants; certain exceptions exist, namely, those in which the corresponding series in x proceeds according to powers of the square root of x. The exact statement of the results will be found italicized on pages 338, 339, 347, 349.

In carrying out the discussion, for both the real and the imaginary cases, we find it convenient to represent our curves, not in cartesian coördinates x, y, but in minimal coördinates u, v, where

$$u = x + iy, \qquad v = x - iy.$$

For a real point x and y are both real, while u and v are conjugate complex quantities. The general analytic element is then represented by writing v as a series which may proceed according to integral powers of either u or some root of u, say $\sqrt[p]{u}$. The integer p is then an obvious arithmetic invariant. When $p = 1$, the element is regular; when $p > 1$ the element is irregular.

We shall throughout this paper write our element in the form

$$v = \alpha_q u^{q/p} + \alpha_{q+1} u^{(q+1)/p} + \alpha_{q+2} u^{(q+2)/p} + \cdots,$$

where we assume $q \geqq p$. This is fair since, if $q < p$, we could interchange the coördinates u and v, which would render $q > p$. We always assume that the leading coefficient α_q does not vanish.

The integer q is a second arithmetic invariant. All elements obtained by taking arbitrary values of the coefficients in the above equation, but fixing the values of both p and q, we shall define as forming a single *species*, the species (p, q).

Just as projective geometry may be discussed either for the real plane or the complex plane, so we may have conformal geometry either for the real or the complex domain. In the real plane we have ∞^2 points defined by two real variables x and y or one complex combination $z = x + iy$: this is the usual gaussian plane. In the complex plane we have ∞^4 points defined by two independent complex coördinates x and y, or by the two linear combinations

$$u = x + iy, \qquad v = x - iy,$$

which are no longer necessarily conjugate but are completely independent complex numbers.

In passing from real to complex projective geometry (of the plane) we also extend our projective group, so that it contains 16 instead of 8 parameters. So in complex conformal geometry we have a larger group of transformations than the usual conformal group. The general conformal transformation (real or imaginary) is obtained by writing U as a power series in u, and V as a power series in v, the coefficients in the series being independent complex quantities; only when these coefficients are conjugate will the transformation be real. In cartesian coördinates our larger group is found in the form

$$X = \phi(x, y), \qquad Y = \psi(x, y),$$

where ϕ and ψ are any power series in two variables with real or imaginary coefficients satisfying the Cauchy-Riemann equations.

§ 2. General Method and Results

In order to find the conformal invariants of the general analytic element of species (p, q), namely

$$v = \alpha_q u^{q/p} + \alpha_{q+1} u^{(q+1)/p} + \alpha_{q+2} u^{(q+2)/p} + \cdots \qquad (\alpha_q \neq 0), \tag{1}$$

we inquire when this element is equivalent to some other element of the same species

$$V = A_q U^{q/p} + A_{q+1} U^{(q+1)/p} + A_{q+2} U^{(q+2)/p} + \cdots. \qquad (A_q \neq 0). \tag{2}$$

Equivalence means that the first equation can be converted into the second by a transformation of the form

$$\begin{aligned} U &= a_1' u + a_2' u^2 + \cdots, \\ V &= b_1' v + b_2' v^2 + \cdots, \end{aligned} \tag{3}$$

in which neither a_1' nor b_1' is to vanish since the transformation is to be regular at the origin. We shall find it convenient to write our transformation in the less symmetric form

$$\begin{aligned} u &= a_1 U + a_2 U^2 + \cdots && (a_1 \neq 0), \\ V &= b_1 v + b_2 v^2 + \cdots && (b_1 \neq 0), \end{aligned} \tag{4}$$

where the a's are of course the coefficients of the series obtained by reverting the first series in (3).

To express the fact that (4) converts (1) into (2) we may eliminate the three quantities u, v, V from the four equations (1), (2), (4). The result is an

equation in U which must hold identically, so that we may equate coefficients of like powers. In order to avoid fractional exponents we make the substitution

$$U = t^p; \tag{5}$$

our *fundamental condition of equivalence* then takes the form

$$A_q t^q + A_{q+1} t^{q+1} + \cdots = b_1 P + b_2 P^2 + \cdots, \tag{6}$$

where

$$P \equiv \alpha_q t^q (a_1 + a_2 t^p + a_3 t^{2p} + \cdots)^{q/p} + \alpha_{q+1} t^{q+1} (a_1 + a_2 t^p + a_3 t^{2p} + \cdots)^{(q+1)/p} + \cdots, \tag{6'}$$

which can obviously be developed as a series in integral powers of t, beginning with t^q, since a_1 does not vanish.

Equating coefficients of like powers of t in (6), we obtain an infinite set of equations involving the constants α and A of the two curves and the constants a and b of the transformation. Two given curves (1) and (2) will be equivalent provided this infinite set of equations can be solved for the a's and b's, subject of course to the essential restriction $a_1 \neq 0$, $b_1 \neq 0$.

If for all values of the α's and A's, the a's and b's can be found, then the curves are always equivalent and no invariants exist. Otherwise certain conditions must be imposed on the α's and A's in order to render the equations consistent, and this indicates the existence of invariants. Which of these possibilities actually occurs we shall find depends essentially on the values of integers p and q determining the species. Some species have invariants, others have not.

The integers p and q are obviously of invariant character under the conformal group. In some types other such *arithmetic invariants* exist. But our main question is to find *absolute* or *differential invariants*, that is, expressions depending on a finite number of the coefficients of the curve, say α_q up to α_{q+r}, which are unchanged by conformal transformations, and thus are converted into expressions of the same form in the new coefficients A_q up to A_{q+r}. The principal result obtained (proofs are given later) is as follows:

The regular species $(1,1), (1,2), (1,3), \cdots$, *have no differential invariants. But with the single exception of the species* $(2,2)$, *all the irregular species have differential invariants.*

When $p = 1$, there are no invariants; also when $p = 2$ and $q = 2$ there are no invariants; but in all other cases the analytic element (1) has invariants.

We shall divide our discussion into three parts, namely, $p = 1$, $p = 2$, $p > 2$. The first and third cases are very simple; but the second is somewhat complicated, since we shall find that distinct discussions are required for three subcases, namely, $q = 2$, $q = 3$, $q > 3$.

We shall in each case either prove that no invariants exist, or else give the explicit expression of the first invariant (that is, invariant of lowest order, where order refers to the subscript $q + r$). A few specimens of additional invariants are given. The number of additional invariants is always infinite.

We restate our notation. Any analytic element may be written in minimal coördinates $u = x + iy$, $v = x - iy$, as follows

$$v = \alpha_q u^{q/p} + \alpha_{q+1} u^{(q+1)/p} + \alpha_{q+2} u^{(q+2)/p} + \cdots$$

where the coefficients α are any complex numbers (making the series convergent), the first coefficient α_q being different from zero; p and q are positive integers and $q \geqq p$. The integers p and q are then arithmetic invariants under the conformal group. All elements with the same p and q we speak of as forming the *species* (p, q). We shall call p the *index* of the element and q the *rank* of the element. If $p = 1$ the element is *regular*; if $p > 1$ it is *irregular*. If $q = p$ the tangent line (at the origin or base point) is not minimal; while if $q > p$ the tangent is minimal. Of course *all real elements are included in the former category* $q = p$.

Absolute or differential invariants, that is, functions of the coefficients unaltered by the conformal group, exist for all irregular species except in the case of species $(2, 2)$. *The order of the lowest invariant is* $q + 2$ *when* $p > 2$. *If however* $p = 2$ *the order is* $q + 3$ *when* $q > 3$, *and* $q + 5$ *when* $q = 3$. *Every species that has invariants has actually an infinite number of invariants.*

The following table exhibits some of the results in detail:

	1	2	3	4	5	6
1	*	*	*	*	*	*
2		†	8	7	8	9
3			5	6	7	8
4				6	7	8
5					7	8
6						8

Here the species is determined by the value of p in the left column and the value of q in the top row. In the body of the table we then find the order of the lowest absolute invariant. The asterisk indicates that the corresponding species has no absolute invariant and further that all members of that species are conformally equivalent. The dagger, in the case of the species $(2, 2)$, indicates that there are no absolute invariants but that the members of the species are not all conformally equivalent (not even in the formal sense): there exists a certain arithmetic invariant, hence there is a division of this species into an infinite number of conformally distinct sub-

species. See § 6 below. We recall that all *real* curves are included in the species $(1,1)$, $(2,2)$, $(3,3)$, etc.; all *real regular* curves are included in $(1,1)$.

When $p > 2$, the lowest invariant is a function of the first *three* coefficients α_q, α_{q+1}, α_{q+2}. When $p = 2$ and $q > 3$, the lowest invariant involves the first *four* coefficients. Finally when $p = 2$ and $q = 3$, the result involves *six* coefficients.

§ 3. Discussion for $p > 2$

This is extremely simple, for we find that the first three equations obtained by the process already described, equating coefficients in the fundamental equation (6), involve only two of the transformation constants. The equations are in fact

$$A_q = \alpha_q b_1 a^{q/2}, \qquad A_{q+1} = \alpha_{q+1} b_1 a^{(q+1)/2}, \qquad A_{q+2} = \alpha_{q+2} b_1 a^{(q+2)/2}.$$

Eliminating a_1 and b_1, we find

$$\frac{\alpha_q \alpha_{q+2}}{\alpha_{q+1}^2} = \frac{A_q A_{q+2}}{A_{q+1}^2}.$$

Hence when the index p of the analytic element (1) *is greater than two the invariant of lowest order is*

$$\frac{\alpha_q \alpha_{q+2}}{\alpha_{q+1}^2};$$

the order $= q + 2$, weight $= 2q + 2$, degree $= 2$.

§ 4. Discussion for $p = 2$, $q > 3$

In this case the first two equations

$$A_q = \alpha_q b_1 a^{q/2}, \qquad A_{q+1} = \alpha_{q+1} b_1 a^{(q+1)/2}$$

determine as before the values of a_1 and b_1. The next two equations

$$A_{q+2} = \alpha_{q+2} b_1 a_1^{(q+2)/2} + \frac{q}{2} \alpha_q b_1 a^{(q+2)/2} a_2,$$

$$A_{q+3} = \alpha_{q+3} b_1 a^{(q+3)/2} + \frac{q}{2} \alpha_{q+1} b_1 a^{(q-1)/2} a_2$$

bring in only a_2, not b_2. Eliminating a_1, b_1, and a_2 from our four equations and separating the coefficients α and A, we find

When p equals two and q is greater than three, the irregular element of species (p, q) has the following invariant of lowest order

$$\frac{\alpha_q (\alpha_q \alpha_{q+3} - \alpha_{q+1} \alpha_{q+2})}{\alpha_{q+1}^3}.$$

Order $= q + 3$, weight $= 3q + 3$, degree $= 3$.

§ 5. Discussion for $p = 2, q = 3$

We shall now show that an ordinary cusp with a minimal tangent line has a differential invariant. This is the species defined by $p = 2, q = 3$; that is

$$v = \alpha_3 u^3 + \alpha_4 u^4 + \alpha_5 u^5 + \cdots .$$

We shall need the first six equations obtained in the usual way from the equivalence identity. They are

$$b_1 \alpha_3 = A_3 a_1^{3/2}, \qquad b_1 \alpha_4 = A_4 a_1^{4/2}, \qquad b_1 \alpha_5 = \tfrac{5}{2} A_3 a_1^{1/2} a_2 + A_5 a_1^{5/2},$$

$$b_1 \alpha_6 + b_2 \alpha_3^2 = 2A_4 a_1 a_2 + A_6 a_1^{6/2},$$

$$b_1 \alpha_7 + 2b_2 \alpha_3 \alpha_4 = A_3 (\tfrac{3}{8} a_1^{-1/2} a_2^2 + \tfrac{3}{2} a_1^{1/2} a_3) + \tfrac{5}{2} A_5 a_1^{3/2} a_2 + A_7 a_1^{7/2},$$

$$b_1 \alpha_8 + b_2 (\alpha_4^2 + 2\alpha_3 \alpha_5) = A_4 (a_2^2 + 2a_1 a_3) + 3A_6 a_1^2 a_2 + A_8 a_1^4 .$$

We are now trying to prove the existence, not the non-existence of invariants; hence it is necessary to have these equations in full, not merely in leading terms, as will be the case in § 6.

The first two equations involve two transformation coefficients a_1, b_1; the next two bring in two new coefficients a_2, b_2; but the following two bring in merely a_3, since b_3 does not as yet appear. Hence it is possible to eliminate the transformation coefficients, and obtain a relation between the coefficients $\alpha_3 \cdots \alpha_8$ of the old curve, and the coefficients $A_3 \cdots A_8$ of the new curve. Carrying out the elimination we find that the two sets of coefficients may be separated,* the result being the expression

$$\frac{\alpha_3^2 (6\alpha_3 \alpha_4 \alpha_7 - 3\alpha_4^2 \alpha_6 - 22\alpha_4 \alpha_5^2 - 9\alpha_3^2 \alpha_8 + 18\alpha_3 \alpha_5 \alpha_6) + 4\alpha_3 \alpha_4^3 \alpha_5}{\alpha_4^5}$$

equal to the same expression in the A's.

Hence this expression is an absolute invariant of our element of species ($p = 2, q = 3$). It is of order 8, weight 20, and degree 5.

In cartesian coördinates such an irregularity is represented by

$$y = ix + c_3 x^3 + c_4 x^4 + \cdots .$$

It is not usually possible to reduce this conformally to the simple cubic cusp

$$y = ix + x^3$$

with minimal tangent line. A necessary condition is that the absolute invariant (a certain combination of c_3, c_5, c_6, c_7, c_8) shall have a special numerical value.†

* This calculation was carried out by Mr. J. A. Northcott in my seminar at Columbia University, 1913.

† The special case where α_4 vanishes of course defines an invariant subspecies under the given species, the first absolute invariant then becoming infinite. An invariant of higher order may then be found for this special case (see § 8).

§ 6. Discussion for $p = 2$, $q = 2$

We now prove that the species $p = 2$, $q = 2$, namely

$$v = \alpha_2 u^{\frac{1}{2}} + \alpha_3 u^{\frac{1}{3}} + \alpha_4 u^{\frac{1}{4}} + \cdots,$$

has no differential invariants, making first the assumption $\alpha_3 \neq 0$.

The equivalence identity takes the form

$$b_1 P + b_2 P^2 + \cdots = A_2 t^2 + A_3 t^3 + \cdots,$$

where

$$P \equiv P_2 t^2 + P_3 t^3 + \cdots,$$

the coefficients here being

$$P_2 = \alpha_2 a_1, \qquad P_3 = \alpha_3 a_1^{3/2}, \qquad P_4 = \alpha_2 a_2 + \cdots,$$

$$P_5 = \tfrac{3}{2}\alpha_3 a_1^{1/2} a_2 + \cdots, \qquad P_6 = \alpha_2 a_3 + \cdots, \qquad P_7 = \tfrac{3}{2}\alpha_3 a_1^{1/2} a_3 + \cdots,$$

. . . . . .

Only the leading terms, that is terms involving the a of highest subscript, are written out since the other terms are unnecessary for the present purpose.

Equating coefficients of powers of t in the identity, we have an infinite system of equations involving the constants α and A of the two arbitrary curves, and the constants a and b of the transformation. We must show that these equations can always be solved for the a's and b's, with neither a_1 nor b_1 vanishing.

The first pair of equations, arising from t^2 and t^3, is

$$\alpha_2 a_1 b_1 = A_2,$$

$$\alpha_3 a_1^{3/2} b_1 = A_3,$$

giving the unique solution

$$a_1 = \left(\frac{A_3\,\alpha_2}{A_2\,\alpha_3}\right)^2, \qquad b_1 = \frac{A_2^3\,\alpha_3^2}{A_3^3\,\alpha_2^2},$$

finite and different from zero, since the first two terms in the series for the curves are assumed to be actually present.

The next pair of equations, arising from t^4 and t^5, are linear in two new unknowns, a_2 and b_2, the determinant of the left hand members being

$$\begin{vmatrix} \alpha_2 b_1 & \alpha_2 a_1 \\ \frac{3}{2}\alpha_3 a_1^{1/2} b_1 & 2\alpha_2 \alpha_3 a^{\frac{3}{2}} \end{vmatrix}.$$

To show that this does not vanish, we may cancel out the non-vanishing factors, α_2, α_3, a_1, b_1; the remaining purely numerical factor is found to be

$$\begin{vmatrix} 1 & 1 \\ \frac{3}{2} & 2 \end{vmatrix} = \tfrac{1}{2}.$$

Similarly the next pair of equations bring in linearly two new unknowns, a_3 and b_3, and the numerical factor in their determinant is

$$\begin{vmatrix} 1 & 1 \\ \frac{3}{2} & 3 \end{vmatrix} = \frac{3}{2}.$$

It is easily verified that the nth pair of equations, arising from the terms t^{2n} and t^{2n+1}, always bring in just two new unknowns, a_n and b_n, and that these equations admit a unique solution, since the determinant, factorable into powers of α_2, α_3, a_1, b_1, together with the number

$$\begin{vmatrix} 1 & 1 \\ \frac{3}{2} & n \end{vmatrix} = n - \frac{3}{2},$$

cannot vanish.

It follows that the coefficients of the transformation cannot be eliminated from the first $2n$ equations, no matter how great we take n. Hence there cannot be any invariant relations involving a finite number of coefficients of the two curves. It follows that our curve can be reduced formally to the normal form $v = u^{\frac{3}{2}}$, and that this reduction is unique.

So far we have used the assumption $\alpha_3 \neq 0$, that is, that the term $u^{\frac{3}{2}}$ is actually present. Since we are discussing the class $p = 2, q = 2$ the term $u^{\frac{3}{2}}$ must of course be present; that is $\alpha_2 \neq 0$. Let us now consider what happens if any number of successive terms after the first are absent.

Let $k = 2m + 1$ denote the subscript of the first term with a fractional exponent actually appearing in the series for the curve. The equation of the curve is thus of the form

$$v = \alpha_2 u + \alpha_4 u^2 + \cdots + \alpha_{2m} u^m + \alpha_{2m+1} u^{m+\frac{1}{2}} + \cdots.$$

The polynomial of mth degree in the first part of this development can always be transformed into a linear term. Hence we may, without loss of generality, take our curve in the reduced form

$$v = \alpha_2 u^{\frac{3}{2}} + \alpha_k u^{k/2} + \alpha_{k+1} u^{(k+1)/2} + \cdots,$$

where $k = 2m + 1$ is odd, and $\alpha_2 \neq 0$, $\alpha_k \neq 0$.

We must show that such a curve can always be transformed into an arbitrary curve of the same type

$$V = A_2 U^{\frac{3}{2}} + A_k U^{k/2} + A_{k+1} U^{(k+1)/2} + \cdots.$$

The equivalence identity is

$$b_1 P + b_2 P^2 + \cdots = A_2 t^2 + A_k t^k + \cdots,$$

where

$$P \equiv \alpha_2 a_1 t^2 + (\alpha_2 a_2 + \cdots) t^4 + (\alpha_2 a_3 + \cdots) t^6 + \cdots + \alpha_k a_1^{k/2} t^k + \left(\frac{k}{2} \alpha_k a^{(k-2)/2} a_2 + \cdots\right) t^{k+2} + \left(\frac{k}{2} \alpha_k a_1^{(k-2)/2} a_3 + \cdots\right) \cdot t^{k+4} + \cdots.$$

Here again only the leading terms with respect to the coefficients a are needed.

Writing out the system of equations obtained by equating coefficients of like powers of t, we find that the first two unknowns a_1, b_1, are not, as in the former case, given by the terms in t^2 and t^4, but rather by those in t^2 and t^k. These equations are not linear but give non-vanishing values for a_1 and b_1.

The next pair of equations, linear in a_2 and b_2, are given by the terms t^4 and t^{k+2}. In general the unknowns a_n and b_n are determined by a pair of linear equations, arising from the terms in t^{2n} and $t^{k+2(n-1)}$. The determinant involves as a factor, in addition to powers of a_1, b_1, α_2, α_k which do not vanish, only the number

$$\begin{vmatrix} 1 & 1 \\ \frac{k}{2} & n \end{vmatrix} = n - \frac{k}{2}$$

which can never vanish since k is odd.

Therefore the type considered has no absolute invariants. The only distinctions are those that arise from the variation of the integer $k = 2m + 1$. Any curve can be converted into any other having the same value of m.

Every irregular element of species $p = 2$, $q = 2$ can be reduced to the normal form*

$$v = u + u^{m+\frac{1}{2}}$$

where m is a positive integer $1, 2, 3, \cdots$. There are no differential invariants. The only arithmetic invariant is the integer m.

The transformation converting the curve into the normal form is determinate, but not always uniquely determinate. This is due to the fact that the first pair of equations, that is those giving a_1 and b_1, are non-linear, at least with respect to a_1. The equations are

$$\alpha_2 a_1 b_1 = A_2, \qquad \alpha_k a_1^{m+\frac{1}{2}} b_1 = A_k,$$

and have $2m - 1$ solutions

$$a_1 = \left(\frac{A_k \alpha_2}{A_2 \alpha_k}\right)^{\frac{2}{2m-1}}, \qquad b_1 = \frac{A_2}{\alpha_2} a_1.$$

Only when $m = 1$ is the required reducing transformation unique.

In particular the number of transformations converting one of our curves into itself is $2m - 1$. The only transformation of $v = u + u^{\frac{3}{2}}$ into itself is identity.†

* The question of convergency is left open for future discussion. This does not, of course, affect results as to the existence or non-existence of invariants (that is, differential invariants of finite order). See paper on *Conformal geometry* cited in § 1.

† We refer throughout only to direct conformal transformations. It may be shown that there is also a unique reverse conformal transformation which converts this curve into itself. The general reverse (or improper) conformal transformation is represented in minimal coordinates by writing U as a series in v, and V as a series in u.

If we pass from the notation of minimal coördinates u, v to that of cartesian coördinates x, y we may express our results as follows:

The class of curves (analytic elements) defined by $p = 2, q = 2$, that is curves having a cartesian equation of the form*

$$y = c_3 x^{\frac{3}{2}} + c_4 x^{\frac{4}{2}} + c_5 x^{\frac{5}{2}} + \cdots,$$

has no differential invariants under the conformal group. Such a curve can be reduced† conformally to the normal form

$$y = x^{m+\frac{1}{2}} \qquad (m = 1, 2, 3, \cdots).$$

The only conformal distinctions are with respect to the arithmetic invariant m. The number of direct conformal transformations of this normal form into itself is $2m - 1$.

Geometrically, the type of irregularity defined by $p = 2$, $q = 2$ is that of an ordinary cusp, with a non-minimal tangent. Variation of m signifies a difference in the (fractional) order of contact between the curve and its tangent line. The case $m = 1$, leads to the simplest cusp, represented by the semi-cubic parabola

$$y = x^{\frac{3}{2}}.$$

If we apply a conformal transformation to this, we obtain an analytic curve with a cusp of the same kind

$$y = c_3 x^{\frac{3}{2}} + c_4 x^{\frac{4}{2}} + \cdots \qquad (c_3 \neq 0).$$

Since $m = 1$, the transformation is uniquely determined by the curve. Hence have this peculiar result:

If we know the effect of a conformal transformation (of course assumed regular in the neighborhood of the cusp) on the points of the semi-cubic parabola (or any conformally equivalent curve), that is if we know the equation of the new or transformed curve, the transformation is uniquely determined.‡

This is apparently the simplest possible way of specifying (completely determining) a conformal transformation by means of curves.

7. Discussion for $p = 1$. Regular Elements

In this simple case there are no differential invariants. This is well-known for the species $(1, 1)$, that is, when the initial tangent is non-minimal. It is true also when the tangent is minimal, that is when q is greater than unity, as we shall show below.

* We take the tangent as axis of x and the normal as axis of y. This is legitimate since the tangent is non-minimal.

† At least in the formal sense.

‡ Contrast this with the familiar theorem: If we know the effect of a conformal transformation on the axis of reals, $y = 0$, the transformation is not determined but involves an infinitude of arbitrary constants.

Regular elements have no differential invariants under the conformal group. All the elements of the same species $(p = 1, q = k)$, *that is,*

$$v = \alpha_k u^k + \alpha_{k+1} u^{k+1} + \cdots,$$

are equivalent to each other, being reducible to the normal form

$$V = U^k;$$

so that arithmetic invariant k *is the only invariant.*

To prove this we may of course use the general identity (6). But we find it more convenient here to use the following simpler method, which settles at the same time all convergency questions.

To show that the given curve can be reduced to the normal form stated, it is sufficient to assume that v is unaltered, and to seek the proper transformation of u. Using

$$U = g(u), \qquad V = v,$$

we find that the function g must satisfy the relation

$$g(u) = u \sqrt[k]{\alpha_k + \alpha_{k+1} u + \alpha_{k+2} u^2 + \cdots}.$$

Since $\alpha_k \neq 0$, this formula defines an integral power series and the coefficient of u in the result does not vanish. Hence our conformal transformation exists.

The number of possible transformations is infinite, depending on an infinite set of arbitrary coefficients. To show this, it is sufficient to seek the most general transformation

$$U = a_1 u + a_2 u^2 + \cdots,$$

$$V = b_1 v + b_2 v^2 + \cdots,$$

which converts the canonical curve $V = U^k$ into itself. The requisite condition is

$$a_1 + a_2 u + a_3 u^2 + \cdots \equiv \sqrt[k]{b_1 + b_2 u^k + b_3 u^{2k} + \cdots},$$

which can be satisfied by taking the b's arbitrarily with $b_1 \neq 0$; then finding the a's, which are (up to a k-th root of unity) determined.*

It is thus seen that any curve of species $(1, k)$ *can be converted conformally into any curve of the same species in an infinity* (∞^∞) *of ways. If we take not only the two curves at random but also an arbitrary regular analytic correspondence between their points (the origin going into the origin) the conformal transformation exists and is uniquely determined.*

Of course in the real conformal plane the only regular elements are those

* On the other hand it is *not* possible, when k exceeds unity, to take the a's arbitrarily and then find the b's.

of species $(1, 1)$, that is to say k is then unity. The results are then well-known.

The natural conformal classification of regular analytic elements in the complex plane, expressed in the usual cartesian apparatus, is as follows:

Every REGULAR *analytic element in the complex plane can be reduced to one of these three forms:*

(I) $$y = 0,$$

(II) $$y = ix + x^k \qquad (k = 2, 3, 4, \cdots),$$

(III) $$y = ix.$$

The first is of course the most general form: it contains all elements whose tangent line is non-minimal (including as a special case all real elements.) The second includes all curved elements whose tangent line is minimal, the order of contact with the minimal line being $k - 1$. The third is the most special form; it includes merely the minimal straight lines of the plane.

In a previous paper,* the author showed that regular elements may be completely characterized, with respect to conformal equivalence, by the value of the limit of the ratio of the arc to the chord running from the given base point to a neighboring point. For ordinary curves, that is, class (I), this limit is of course unity. But for class (II) the result is

$$L = \text{limit } \frac{\text{arc}}{\text{chord}} = \frac{2\sqrt{k}}{k+1},$$

which is, for example, correct to two decimal places, .94 for $k = 2$, .86 for $k = 3$, .80 for $k = 4$, .74 for $k = 5$. Hence the value of L determines the value of the integer k, so that two elements having the same L will be conformally equivalent. Finally, for class (III) both arc and chord vanish identically, hence the limit L does not exist (is indeterminate).

§ 8. SOME ADDITIONAL INVARIANTS

For each species (p, q) we have either shown that no invariants exist or we have obtained the explicit expression of the first absolute invariant (differential invariant of lowest order). Every species that has an invariant (that is every irregular species except $(2, 2)$) has in fact an infinite number of invariants. We shall here give some examples of the invariants of higher order.

Consider first the species $(3, 3)$, that is

(7) $$v = \alpha_3 u^3 + \alpha_4 u^4 + \alpha_5 u^5 + \cdots \qquad (\alpha_3 \neq 0).$$

Here the first invariant, obtained by elimination from three equations, is, in accordance with the discussion for $p > 2$ given in § 3,

* *The ratio of the arc to the chord of an analytic curve need not approach unity.* Bulletin of the American Mathematical Society, vol. 20 (1914), pp. 524-531.

$$\frac{\alpha_3\,\alpha_5}{\alpha_4}.$$

Assuming that α_4 does not vanish, we find that we can also eliminate the constants of the transformation from the next three equations, the result being an invariant of order 8. Continuing in this way by grouping the equivalence equations into sets of three we find this result:

The general case of species (3, 3), *that is form* (7) *with* $\alpha_4 \neq 0$, *has absolute invariants of order* 5, 8, 11, 14, *etc.*

In the more special case defined by $\alpha_4 = 0$, $\alpha_5 \neq 0$, we find that there still exist an infinite number of invariants, the orders now being 7, 10, 13, 16, etc.

For larger values of p, the structure of a certain first succession of $p-2$ invariants is very simple. Thus for $p = 5$, the first three invariants are

$$\frac{\alpha_q\,\alpha_{q+2}}{\alpha_{q+1}^2}, \qquad \frac{\alpha_q\,\alpha_{q+3}}{\alpha_{q+1}\,\alpha_{q+2}}, \qquad \frac{\alpha_q\,\alpha_{q+4}}{\alpha_{q+1}\,\alpha_{q+3}}.$$

The next higher invariants are more complicated.

For the species (3, 4), we have first a simple invariant of order 6, and second a complicated invariant of order 9. The expressions are

$$\frac{\alpha_4\,\alpha_6}{\alpha_5}, \qquad \frac{\alpha_4^2\,(5\alpha_5^2\,\alpha_7 - 4\alpha_4\,\alpha_5\,\alpha_8 + 2\alpha_4^2\,\alpha_9 - 3\alpha_4\,\alpha_6\,\alpha_7)}{\alpha_5^5}.$$

For the species (4, 5) the first three invariants, of orders 7, 8, and 11, are

$$\frac{\alpha_5\,\alpha_7}{\alpha_6^2}, \qquad \frac{\alpha_6\,\alpha_8}{\alpha_7^2}, \qquad \frac{12\alpha_5^3\,\alpha_6^2\,\alpha_9 - 7\alpha_5^4\,\alpha_7\,\alpha_9 - 10\alpha_5^4\,\alpha_6\,\alpha_{10} + 5\alpha_5^5\,\alpha_{11}}{\alpha_6^6}.$$

In our discussion of lowest invariants the most complicated species was (2, 3), since here it was necessary to use six equations before elimination was possible. Grouping the next equations of equivalence into sets of six, we find that new eliminations are always possible. Hence

The species (2, 3) *has an infinite number of invariants, the orders being* 8, 14, 20, 26, *etc.*

This is of course in the general case of that species. It is assumed that not only the first term $\alpha_3\,u^{\frac{3}{2}}$ is present, but also the second term $\alpha_4\,u^{\frac{4}{2}}$. Separate discussions are necessary if certain terms are absent.

For the arbitrary species (p, q) with a certain number of terms, after the first, absent, we state only this result. If the curve is of the form

$$v = \alpha_q\,u^{q/p} + \alpha_{q+k}\,u^{(q+k)/p} + \alpha_{q+k+1}\,u^{(q+k+1)/p} + \cdots,$$

where $p > 2$ and $k < p-1$, the first invariant is

$$\frac{\alpha_q\,\alpha_{q+k+1}^k}{\alpha_{q+k}^{k+1}}.$$

For the real species (p, p) it is worth while to give the following more detailed results.

The case represented by

$$v = \alpha_q u^{p/p} + \alpha_{q+k} u^{(p+k)/p} + \cdots,$$

when $k < p - 1$ has a first invariant of order $p + k + 1$ and also, provided $k < p - 2$, a second invariant of order $2p - 1$. When $k = p - 1$, the first invariant is of order $2p + 1$.

If still more terms in the power series are absent, say if $\alpha_{p+1} = 0$, $\alpha_{p+2} = 0$, and so on, until $\alpha_{lp+k} \neq 0$, so that the element (p, p) is of the special form

$$v = \alpha_q u^{p/p} + \alpha_{lq+k} u^{(lp+k)/k} + \cdots,$$

then when $k < p - 1$ the first order is $lp + k + 1$; and when $k = p - 1$, assuming $p > 2$, the first order is $(l + 1)p + 1$.

§ 9. Real Elements

For real elements under the group of real conformal transformations we may, in conclusion, state our principal results in the usual cartesian coördinates as follows, taking the base point as origin and the initial tangent of the element as axis of x.

Regular real elements are conformally reducible to $y = 0$.

Irregular real elements of index two, that is

$$y = c_3 x^{\frac{3}{2}} + c_4 x^{\frac{4}{2}} + \cdots$$

have no differential invariants, and may be reduced (formally) to

$$y = x^{m+\frac{1}{2}}, \qquad (m = 1, 2, 3, \cdots).$$

Irregular real elements of index greater than two always have an infinite number of differential invariants under the real conformal group.

A direct discussion of the real theory in cartesian coördinates will appear in another paper. It should be observed that our classification into species (p, q) is based on the minimal notation u, v. If we pass to the cartesian notation x, y, the fractional exponents in the leading term are usually different from q/p. The index p remains as denominator, but the new numerator has no intrinsic significance. A new arithmetic invariant is then obtained which has such significance.

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EXTENSIONS OF DESCARTES' RULE OF SIGNS CONNECTED WITH A PROBLEM SUGGESTED BY LAGUERRE*

BY

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1. A Problem Proposed by Laguerre

As one of the first applications of his proof of Descartes' Rule of Signs in a form admitting immediate extension to power series, Laguerre† considered certain properties of an infinite rectangular array formed from the coefficients of the developments in descending powers of x of the expressions

$$\frac{f(x)}{(x-1)^i} \qquad (i = 1, 2, \cdots),$$

where $f(x)$ is a given polynomial of degree n, and $f(1) \neq 0$. Thus if we write, with a notation differing slightly from Laguerre's,

$$\frac{f(x)}{(x-1)^i} = \sum_{j=0}^{\infty} a_{ij}\, x^{n-i-j},$$

and form the array

$$\begin{array}{cccccc} a_{10} & a_{11} & a_{12} & \cdot & \cdot & \cdot \\ a_{20} & a_{21} & a_{22} & \cdot & \cdot & \cdot \\ a_{30} & a_{31} & a_{32} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array} \tag{1}$$

Laguerre has proved that a superior limit for the number of roots of $f(x)$ in the interval $[1, +\infty]$ is given by the number of variations of sign in the sequence consisting of any horizontal line of (1). If the latter number is greater than the former, the difference is an even number. Laguerre also proves that another type of sequence having the same property is obtained by proceeding along any horizontal line until we reach a term a_{ij} for which $i + j$ is at our choice, subject to the inequality $i + j \geqq n + 1$, and then taking a path diagonally upward to the right. These results we generalize somewhat in § 3.

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† *Oeuvres*, vol. 1, pp. 1–25.

Laguerre notes that the number v of variations of sign in a row of (1) either decreases or remains stationary as we go down in the array, and hence must have a minimum greater than or equal to the number m of roots of $f(x)$ in $[1, +\infty]$. He remarks on the importance of determining this minimum, but considers the problem one of great difficulty.* This problem remained unsolved until very recently, when it was proved by Fekete and Pólya† that if we go sufficiently far down in the array (1), we shall have $v = m$.

The proofs of Fekete and Pólya involve rather intricate systems of inequalities, though they have the advantage of extending the results to cases where $f(x)$ is not a polynomial. It is the purpose of the present paper to present a simpler proof modeled on Laguerre's treatment of a similar problem, where he shows that the number of variations of sign presented by the coefficients in the development of $e^{zx}f(x)$ according to ascending powers of x is equal to the number of positive roots of $f(x)$ provided z is sufficiently large.‡ This method assumes that $f(x)$ is a polynomial, and additional considerations would be necessary to extend the proof to other cases.

In addition to this solution of Laguerre's problem, other results of a similar nature connected with arrays analogous to (1) are obtained in the present paper.

2. Definitions for Laguerre Sequences

We shall here slightly generalize the array (1) by considering the expansions

$$\frac{f(x)}{(x-z)^i} = \sum_{j=0}^{\infty} A_{ij}\, x^{n-i-j} \qquad (i = 0, 1, 2, \cdots), \tag{2}$$

where $f(x)$ is the polynomial

$$f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n \qquad (a_0 \neq 0), \tag{3}$$

and z is a positive number not a root§ of $f(x)$. The array to be studied is then

$$\begin{array}{cccccc} A_{00} & A_{01} & A_{02} & \cdot & \cdot & \cdot \\ A_{10} & A_{11} & A_{12} & \cdot & \cdot & \cdot \\ A_{20} & A_{21} & A_{22} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array} \tag{4}$$

We also generalize the sequences mentioned in § 1, and define as a *Laguerre*

* Loc. cit., p. 14.

† *Ueber ein Problem von Laguerre*, Rendiconti del Circolo Matematico di Palermo, vol. 34 (1912), p. 89. The problem is here stated in slightly different form.

‡ *Oeuvres*, vol. 1, pp. 22–25.

§ The following argument can easily be modified so as to include the case $f(z) = 0$, but the extra complications in statement make it hardly worth while here.

Sequence any sequence composed of the elements which lie on a path beginning anywhere in the first column below the first row of (4) and proceeding at each move either horizontally or diagonally upwards to the right until it reaches some row below the first, along which it proceeds indefinitely to the right; i. e., a sequence

$$A_{r_0 0},\ A_{r_1 1},\ A_{r_2 2},\ \cdots, \tag{5}$$

where, for any i, r_{i+1} may either coincide with r_i or be less by unity than r_i, either case being possible at each step, but all the r's are greater than zero. It will be seen that this definition includes as special cases the two kinds of sequences studied by Laguerre. We shall show that every Laguerre Sequence has properties analogous to those mentioned for these two in § 1.

One more designation we shall find useful. A Laguerre Sequence (5) will be said to *enclose* a second Laguerre Sequence

$$A_{s_0 0},\ A_{s_1 1},\ A_{s_2 2},\ \cdots,$$

if for every i we have $r_i \geqq s_i$; i. e., if each member of the first sequence either coincides with or lies below that member of the second sequence which is in the same column of (4).

3. Formulas for the Elements of the Array (4)

The first $n+1$ elements of the first row of (4) are evidently the coefficients of $f(x)$, and the remaining elements of this row are zeroes; i. e., we have

$$\begin{aligned} A_{0j} &= a_j && (j = 0, 1, \cdots, n), \\ &= 0 && (j > n). \end{aligned} \tag{6}$$

Each of the following rows is obtained from its predecessor by the division algorithm. We thus obtain the formulas

$$\begin{aligned} A_{i0} &= A_{00} && (i = 1, 2, \cdots), \\ A_{ij} &= A_{i-1\,j} + zA_{i\,j-1} && (i > 0,\ j > 0). \end{aligned} \tag{7}$$

By repeated use of (7), or more easily by direct multiplication of (3) by the development of $(x-z)^{-i}$ in descending powers of x, we find that for every $j > 0$,

$$\begin{aligned} A_{ij} = A_{0j} + A_{0\,j-1}\, iz + A_{0\,j-2}\frac{i(i+1)}{2}z^2 + \cdots \\ + A_{00}\frac{i(i+1)\cdots(i+j-1)\,z^j}{j!}. \end{aligned} \tag{8}$$

As the terms of (4) on or beyond what Laguerre styles the *principal diagonal* (the diagonal passing through the elements for which $i+j = n+1$) play a

special rôle in what follows, we proceed, for these terms, to throw (8), with the help of (6), into the form

$$A_{ij} = \frac{(i+j-n-1)!}{(i-1)!\,j!} F(z,i,j)\,z^{j-n} \quad (i+j>n,\ i>0), \tag{9}$$

where

$$\begin{aligned} F(z,i,j) &= a_0(i+j-1)(i+j-2)\cdots(i+j-n)z^n \\ &\quad + a_1 j(i+j-2)\cdots(i+j-n)z^{n-1} + \cdots \\ &\quad + a_k j(j-1)\cdots(j-k+1)(i+j-k-1)\cdots(i+j-n)z^{n-k} \\ &\quad + \cdots + a_n j(j-1)\cdots(j-n+1). \end{aligned} \tag{10}$$

Two other expressions for the elements of (4) are of interest. The first is given by the formula

$$A_{ij} = z^{j-n} S_j^{(i-1)} \qquad (i>0,\ j \geqq 0), \tag{11}$$

where $S_j^{(i-1)}$ denotes the numerator of the $(j+1)$th Cesàro mean* of order $i-1$ for the series

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_n + 0 + 0 + \cdots.$$

The other formula, which applies to terms on or beyond the principal diagonal, is

$$A_{ij} = \frac{1}{(i-1)!} \frac{d^{(i-1)}}{dz^{(i-1)}} (z^{i+j-n-1} f(z)) \quad (i>0,\ i+j>n). \tag{12}$$

This includes the well-known result that the elements of the principal diagonal verify the relation

$$A_{i\,n+1-i} = \frac{1}{(i-1)!} \frac{d^{(i-1)}}{dz^{(i-1)}} f(z) \qquad (0<i<n+2).$$

From this latter formula the truth of (12) follows at once if we note that A_{ij} is on the principal diagonal of the array for the polynomial $x^{i+j-n-1} f(x)$.

4. The Number of Variations of Sign in Laguerre Sequences

From formulas (9) and (10) we derive the property expressed in the following theorem.

Theorem 1. *For every $r>0$ an integer $s>n$ can be determined such that every element of the array* (4) *situated in a row below the first and above the rth, and in a column beyond the sth, is of the same sign as $f(z)$.*

This follows immediately from the fact that $F(z,i,j)$, as given by (10), is a polynomial of degree n in j for which the coefficient of j^n is $f(z)$; and

* See Bromwich, *Theory of Infinite Series*, p. 311. Cf. also Fekete and Pólya, loc. cit., p. 101.

by hypothesis, $f(z) \neq 0$. Hence for each $i > 0$ there is an integer s_i such that A_{ij} is of the same sign as $f(z)$ for all $j \geqq s_i$. In the above theorem s can be taken as the greatest of the numbers $s_1, s_2, \cdots, s_{r-2}$.

From Theorem 1 it follows that every Laguerre Sequence terminates in an infinite number of terms of the same sign as $f(z)$, and consequently has only a finite number of variations of sign.

THEOREM 2. *If L_1 and L_2 are two Laguerre Sequences chosen arbitrarily from the array* (4) *subject to the condition that L_2 encloses L_1, and if V_1, V_2 are the numbers of variations of sign presented by L_1, L_2 respectively, then $V_1 - V_2 = 0$ or an even positive integer.*

To prove this theorem we first note that by formula (7) the sequence

$$A_{i\,j-1},\ A_{i-1\,j},\ A_{i-1\,j+1}$$

presents either the same number of variations of sign, or else two more than

$$A_{i\,j-1},\ A_{i\,j},\ A_{i-1\,j+1},$$

provided $A_{i-1\,j+1} \neq 0$. Hence if L_2 and L_1 are the same except that one term A_{ij} of the former is replaced by $A_{i-1\,j}$ in the latter, the conclusion of Theorem 2 must be true.* This will be the case even when $A_{i-1\,j+1} = 0$, for the number of variations of sign in both L_1 and L_2 would be unchanged if $A_{i-1\,j+1}$ were given the sign of the first succeeding term that does not vanish; by Theorem 1 such terms always exist.

If L_1 and L_2 are now taken in more general fashion, but so that they coincide after k terms, where k is any positive integer, the former sequence may be transformed into the latter by a finite number of substitutions of terms A_{ij} for terms $A_{i-1\,j}$, each such substitution leaving the number of variations of sign unchanged, or decreasing it by two.

Finally, if L_1 and L_2 do not coincide after k terms, no matter how large k may be, we use the conclusion of Theorem 1 to replace L_2 by a sequence L_3 which coincides with L_2 until we have entered that portion of the array where all terms are of the same sign, and then proceeds diagonally upward until a term of L_1 is reached, after which it coincides with L_1. Since the conclusion of the previous paragraph holds for L_1 and L_3, and L_2 and L_3 have the same number of variations of sign, we have now established Theorem 2 for all cases.

THEOREM 3. *Let L be an arbitrarily chosen Laguerre Sequence presenting V variations of sign, and let M be the number of real roots of $f(x)$ in the interval $[z, +\infty]$. Then $V - M = 0$ or an even integer.*

Laguerre has proved† this property for every sequence composed of the

* The case where $j = 0$ is easily disposed of, since $A_{i-1\,0} = A_{i0}$.

† Loc. cit., p. 14.

terms of a row of (4). An arbitrary Laguerre Sequence is enclosed by sequences of this type, hence Theorem 3 follows as a consequence of Theorem 2.

5. Proofs of the Existence of Laguerre Sequences for which $V = M$

We now present a proof, based on Laguerre's methods in a similar case (see § 1), that there are sequences for which V, the number of variations of sign, is exactly equal to M, the number of roots of $f(x)$ in $[z, +\infty]$.

Theorem 4. *There exists a positive integer $r > n$, such that for the sequence composed of the $(r+1)$th horizontal row of* (4) *we have $V = M$.*

If we refer to the notation of formulas (9) and (10) we see that V_i, the number of variations of sign in the $(i+1)$th row of (4), is equal to the number of variations of sign in the sequence composed of the values of $F(z, i, j)$ for $j = 0, 1, 2, \cdots$. Make the substitutions

$$\frac{j}{i} = x, \qquad \frac{1}{i} = \omega.$$

Then we have

$$\frac{1}{i^n} F(z, i, j) = \phi(z, x, \omega), \tag{13}$$

where

$$\begin{aligned} \phi(z, x, \omega) &= a_0(1 + x - \omega)(1 + x - 2\omega) \cdots (1 + x - n\omega) z^n + \cdots \\ &+ a_k x(x-\omega)\cdots(x-(k-1)\omega)(1+x-(k+1)\omega)\cdots(1+x-n\omega)z^{n-k}+\cdots \\ &+ a_n x(x - \omega) \cdots (x - (n-1)\omega). \end{aligned} \tag{14}$$

The number V_i is equal to the number of variations of sign in the sequence formed by the values of $\phi(z, x, \omega)$ for $x = 0, \omega, 2\omega, \cdots$, so that $V_i \leqq M_\omega$, where M_ω denotes the number of positive roots of $\phi(z, x, \omega)$ regarded as a polynomial in x.

We now observe that

$$\phi(z, x, 0) = x^n f\left(\frac{1+x}{x} z\right),$$

so that the number M_0 of positive roots of $\phi(z, x, 0)$ regarded as a polynomial in x is equal to M, the number of roots of $f(x)$ in $[z, +\infty]$. We will prove that if ω_1 is a sufficiently small positive number, we must have $M_\omega \leqq M_0$ for all positive values of ω less than ω_1, so that $M_\omega \leqq M$. Since we have seen above that $V_i \leqq M_\omega$, it follows that $V_i \leqq M$ if i is sufficiently large. But from Theorem 3 we have $V_i \geqq M$, and the proof of our theorem is thus completed when we have established the point above mentioned.

We now show that the number ω_1 mentioned in the preceding paragraph may be chosen as follows: Designate by $\Theta(\omega)$ the discriminant of $\phi(z, x, \omega)$ regarded as a polynomial in x; then ω_1 may be taken as a positive number not

greater than $1/n$, and not greater than the smallest positive root of $\Theta(\omega)$. In the notation of the preceding paragraphs, it remains to prove that $M_\omega \leqq M_0$ for all positive $\omega < \omega_1$. Let ω_0 be such a value of ω, and let ω increase steadily from $\omega = 0$ to $\omega = \omega_0$. The n roots of $\phi(z, x, \omega)$ regarded as a polynomial in x are continuous functions of ω. A glance at (14) shows that $x = 0$ can not be a root of $\phi(z, x, \omega)$ for $\omega < \omega_1 \leqq 1/n$. From this it follows that a root which is negative for $\omega = 0$ can not become positive when ω has increased from 0 to ω_0. A root imaginary for $\omega = 0$ cannot become real for $\omega = \omega_0$, since this would imply the existence of a value of ω between 0 and ω_0 for which the root coincides with its conjugate; but this is impossible because the discriminant $\Theta(\omega)$ does not vanish in $[0, \omega_0]$. Thus the only roots of $\phi(z, x, \omega)$ that can be positive real numbers for $\omega = \omega_0$ are those which were real and positive for $\omega = 0$. In other words, we must have $M_{\omega_0} \leqq M_0$. This completes the proof of Theorem 4; we have in fact shown that r may be chosen as any integer $> 1/\omega_1$.

THEOREM 5. *There exists a positive integer $s > n$ such that the "diagonal sequence" beginning with the term A_{s0} and composed of terms on a line parallel to the principal diagonal of* (4), *followed by terms of the second row, has the property that $V = M$.*

The truth of this follows at once from Theorems 1 and 4. Let r be a positive integer chosen in accordance with Theorem 4. We may replace the sequence L_1 of terms of the $(r+1)$th row of (4) by a sequence L_2 which coincides with L_1 until that part of the array is reached where, by Theorem 1, all terms are of the same sign, after which L_2 proceeds diagonally upward to the second row and thereafter remains in that row. For L_2 we also have $V = M$. If L_2 is now replaced by the diagonal sequence L_3 composed of *all* the terms, up to the second row, in the diagonal along part of which L_2 proceeded from the $(r+1)$th row to the second row, followed by terms of the latter row, we see that L_3 encloses L_2, and hence cannot have more variations of sign than L_2. But by Theorem 3 we have $V \geqq M$ for every Laguerre Sequence; hence for L_3 we have $V = M$, and L_3 satisfies the requirements of Theorem 5.

THEOREM 6. *There exists a Laguerre Sequence L for which $V = M$, and such that all sequences which enclose L have the same property, while all other Laguerre Sequences have $V > M$.*

The sequence L here indicated may be obtained by taking a diagonal sequence which satisfies Theorem 5 and performing on it successive substitutions of the type considered in the proof of Theorem 2, i. e., substitutions of a term $A_{i-1\,j}$ for A_{ij} which are such that the new sequence is a Laguerre Sequence. Only such substitutions are, however, to be made as produce at each step a new sequence with V unchanged. When the last possible substitution in a succession of this kind has been made the result will be the sequence

L of Theorem 6.* To prove this we note first that all enclosing sequences can have no more variations of sign than L, by Theorem 2, and no less than M variations of sign, by Theorem 3, so that $V = M$ for every sequence enclosing L. If a sequence (other than L itself) is enclosed by L it is obtained from L by substitutions of terms of the type above considered, and therefore must have more variations of sign than L. The only case remaining is that of a sequence $\overline{L}$ which neither encloses L nor is enclosed by L. The sequences L and $\overline{L}$ must then have one or more terms common. Let these terms be designated, in the order in which they occur in L, by $A_{i_1j_1}, A_{i_2j_2}, \cdots$. The sequence of terms of L from its first term to the term $A_{i_1j_1}$ inclusive we will style the *segment* L_0, the sequence from $A_{i_1j_1}$ to $A_{i_2j_2}$ inclusive the *segment* L_1, etc. If there is a last common term $A_{i_kj_k}$, the *segment* L_k will begin with this term and include all remaining terms of L. We similarly define segments $\overline{L}_0, \overline{L}_1, \cdots$, of the sequence $\overline{L}$. We shall say that L_p encloses $\overline{L}_p$ if for each corresponding pair of terms in the same column of (4) the term of L_p is in the same row or a lower row than that where the term of $\overline{L}_p$ is situated; similarly if $\overline{L}_p$ encloses L_p. Since $\overline{L}$ does not enclose L nor coincide throughout with it, there must be at least one segment $\overline{L}_p$ enclosed by the corresponding L_p and not coincident with the latter. For such a segment $\overline{L}_p$ the number of variations of sign presented by its terms must be greater than that presented by L_p, otherwise the sequence obtained by replacing in L the segment L_p by the segment $\overline{L}_p$ would be a Laguerre Sequence enclosed by L and having no greater number of variations of sign than L; but this we have seen to be impossible. On the other hand no segment $\overline{L}_q$ which encloses L_q can present less variations of sign than L_q, otherwise the substitution in L of $\overline{L}_q$ for L_q would produce a Laguerre Sequence having less variations of sign than L, which is again impossible. Since all segments of $\overline{L}$ must belong to one of the two classes here considered, and the number of variations of sign in a sequence is the sum of the numbers of variations in a complete set of segments, it follows from the above that $\overline{L}$ must have more variations of sign than L. The proof of Theorem 6 is thus completed.

Let us note here that in all the theorems of this paper we may replace the array (4) by one in which the elements below the first row are the Cesàro means $S_j^{(i-1)}$ (see formula (11)). On account of the importance of Cesàro means in certain questions in other fields, it is of interest to find them playing a rôle in the theory of equations.

6. Laguerre Sequences Analogous to Sequences of Sturm's Functions

The terms A_{ij} of the array (4) are polynomials in z. The question at once presents itself as to how the number of variations of sign in a Laguerre

* Our construction process leads, therefore, to a *unique* result.

Sequence varies with z. In particular, since the number of roots in $[z, +\infty]$ remains stationary or decreases as z increases, it might be hoped that the number of variations of sign in a Laguerre Sequence would always tend to decrease as z increases. A simple example will show that this is not true. Let us take $f(x) = x^2 - 3x + 2$; the second row of (4) is then

$$1 \quad z-3 \quad f(z) \quad zf(z) \quad \cdot \quad \cdot \quad \cdot .$$

For $z = 3/2$ this Laguerre Sequence has one variation of sign, but for $z = 5/2$ there are two. Thus there are even sequences of the type mentioned in Theorem 6 for which our supposition is untrue. We can, however, easily establish the truth of the following statement.

THEOREM 7. *For each positive $z = z_0$ not a root of $f(z)$ there exists a positive integer r such that every Laguerre Sequence beginning in a row of* (4) *below the rth has for every $z \geqq z_0$, subject to the restriction $f(z) \neq 0$,* exactly as many variations of sign as $f(x)$ has roots in $[z, +\infty]$.*

To prove this we first note that if $\phi(x)$ is any polynomial of degree m whose *Fourier Sequence,*

$$\phi(x), \qquad \frac{d}{dx}\phi(x), \qquad \cdots, \qquad \frac{d^m}{dx^m}\phi(x)$$

has the number V_z of variations of sign for $x = z$, and if a is such a number (not a root of $\phi(x)$) that $V_a = M_a$, where M_a is the number of roots of $\phi(x)$ in $[a, +\infty]$, then for all $b > a$ such that $\phi(b) \neq 0$ we shall have $V_b = M_b$. This follows from the well-known theorem of Fourier which is expressed by the formula

$$V_a - V_b \geqq M_a - M_b .$$

Since $V_a = M_a$, we must have $V_b \leqq M_b$; but as a corollary of Fourier's theorem $V_b \geqq M_b$; hence $V_b = M_b$.

By formula (12) a diagonal sequence having for $z = z_0$ the property expressed in Theorem 5 is composed of the terms, in reversed order and each multiplied by a positive constant, of the Fourier Sequence for $z^k f(z)$ where k is a positive integer, followed by terms of the same sign as $f(z)$. Hence from the above remark on Fourier Sequences it follows that a diagonal sequence verifying Theorem 5 for $z = z_0$ satisfies the requirements of Theorem 7. From Theorems 2 and 3 the same must be true for every Laguerre Sequence enclosing such a diagonal sequence, i. e., for every sequence of the type mentioned in Theorem 7.

This argument also establishes the truth of the following statement.

THEOREM 8. *Each of the sequences indicated in Theorem 7 has the property*

* This restriction, imposed by the form of our previous results, may easily be removed. In that case Theorem 7 will still be true if $[z, +\infty]$ is defined as not including z.

for the interval $[z_0, +\infty]$ *possessed by sequences of Sturm's functions, that for every subinterval* $[a, b]$ *where* $f(a) \neq 0, f(b) \neq 0$,* *the number of variations of sign for* $z = a$ *diminished by the number for* $z = b$ *is equal to the number of roots of* $f(x)$ *in* $[a, b]$.

7. The Cartesian Multiplier $(x + z)^m$

Fekete and Pólya† have announced without proof that if m is a sufficiently large positive integer, $(x + 1)^m$ will be what I have elsewhere‡ styled a *Cartesian Multiplier* for $f(x)$, i. e., a polynomial such that the number of variations of sign presented by the coefficients in the product of $f(x)$ by the Cartesian Multiplier is equal to the exact number of positive roots of $f(x)$. A proof similar to that of Theorem 4 would establish the following theorem analogous to the theorem of Fekete and Pólya; we shall here, however, give a proof which depends on the preceding theorems of this paper.

Theorem 9. *For each positive number* z *there exists a positive integer* k *such that for all positive integers* m *greater than* k *the polynomial* $(x + z)^m$ *is a Cartesian Multiplier for the given polynomial* $f(x)$.

There is nothing in the proofs of Theorems 1–6 which would prevent our substituting for $f(x)$ a polynomial whose coefficients are functions of z. In particular, let us form the array (4) for the polynomial $f(x - z)$. For our former proviso that $f(z) \neq 0$ we must now substitute $f(0) \neq 0$. One can, however, easily change the following statements so as to include the case $f(0) = 0$.

In the array (4) for $f(x - z)$, the elements in and beyond the principal diagonal are given by the analogue of formula (12),

$$\bar{A}_{ij} = \frac{1}{(i-1)!} \frac{d^{(i-1)}}{dx^{(i-1)}} [x^{i+j-n-1} f(x - z)] \Big|_{z=x} \qquad (i > 0,\ i + j > n). \tag{15}$$

The transformation $x - z = X$ changes this into

$$\bar{A}_{ij} = \frac{1}{(i-1)!} \frac{d^{(i-1)}}{dX^{(i-1)}} [(X + z)^{i+j-n-1} f(X)]_{X=0}. \tag{16}$$

Thus *the terms* $\bar{A}_{ij}$ *of the array* (4) *formed for* $f(x - z)$ *on a diagonal beyond the principal diagonal, i. e., terms for which* $i + j = m + n + 1$, *where* m *is a positive integer, are the coefficients of the powers of* x *in the polynomial*

$$(x + z)^m f(x).$$

* We may include the cases where either a, or b, or both are roots of $f(x)$ by making $[a, b]$ include b in all cases, but a only when $f(a) \neq 0$.

† Loc. cit., p. 120.

‡ *An Extension of Descartes' Rule of Signs*, Mathematische Annalen, vol. 73 (1912), p. 424.

By Theorem 5 applied to $f(x-z)$, the integer k can be determined so that for $m = k+1$ the number of variations of sign in the corresponding diagonal sequence is equal to the number of roots of $f(x-z)$, as a polynomial in x, in $[z, +\infty]$. By Theorem 2 the same must be true for all values of $m > k$. Our proof of Theorem 9 is now complete if we observe first that all the variations of sign occur in the diagonal parts of these sequences, and then note that there is a one-to-one correspondence between the roots of $f(x-z)$ in $[z, +\infty]$, and those of $f(x)$ in $[0, +\infty]$.

We could form an array of the coefficients of $(x+z)^m f(x)$ for $m = 0, 1, \cdots$, and sequences formed according to certain easily inferred laws would have properties corresponding to those for Laguerre Sequences given in Theorems 1–6. The reader will, however, find little difficulty in carrying this out for himself.

ON PARASTROPHIC ALGEBRAS*

BY

JAMES BYRNIE SHAW

The following paper is a consideration of the relation to a given algebra of algebras produced from it by the interchange of certain of the multiplication constants for an assigned basis. In particular the process of multiplying parastrophically is studied somewhat, with some simple applications.

A very simple test is thus furnished for ascertaining whether a given algebra is semi-simple or not, a test not always applicable but very simple when it is applicable. The parastrophic algebra is not an invariant of the given algebra but changes with a change of the basis. However certain expressions furnished by the parastrophic algebra are invariant for the given algebra, at least in a certain defined sense.

The parastrophic algebras are shown to be determinable when an operator ρ' is determined, defined uniquely by ρ. In the important case (there are also others) of a Dedekind algebra in canonical form, the operator ρ' corresponds to the multiplication by a definite number of the algebra. The parastrophic multiplication becomes then the original multiplication not by ρ but by a definite number τ of the algebra determined by ρ.

The results of much of the paper by Frobenius† on hypercomplex numbers are implied in the parastrophic multiplications and arrived at in a very direct and simple manner. The introduction of the numbers κ with their significance indicates the manner of procedure in the reduction of an algebra to its canonical form.‡

1. We assume an algebra with a given multiplication table, the products being linear in the variables implied in the numbers of the algebra. Related to this multiplication will be other multiplication tables which we proceed to define. These are chosen from an infinite variety of other related tables on account of their simplicity. Each table defines an algebra which may or may not agree with the original algebra. Thus let two general numbers of these related algebras be

$$\rho = \sum x_i e_i, \qquad \sigma = \sum y_j e_j,$$

* Read before the Chicago section of the Society, December 29, 1914.

† Sitzungsberichte K. Akademie Berlin, 1903, pp. 504–537, 634–645.

‡ For terms in general see Shaw, *Synopsis of Linear Associative Algebra*, 1907.

wherein the variables x and y run over assigned ranges, not at first specified, and where the units e are any numbers from the algebra, r in number, not subject to any conditional equation of a linear form, such as $\sum a_i e_i = 0$, unless every a vanishes.

We then define the original algebra by the equation of general multiplication

$$M_1 \cdot \rho\sigma = \sum x_i y_j \gamma_{ijk} e_k \qquad (i, j, k = 1, \cdots, r).$$

The constants γ are assigned in some manner, and may or may not run over the same range as x and y. The related algebras that are mentioned here will then be defined by the equations following, and will be named as indicated, the original being the *strophic*:

$$M_2 \cdot \rho\sigma = \sum x_i y_j \gamma_{jik} e_k, \qquad \textit{antistrophic},$$

$$M_3 \cdot \rho\sigma = \sum x_i y_j \gamma_{ikj} e_k, \qquad \textit{preparastrophic},$$

$$M_4 \cdot \rho\sigma = \sum x_i y_j \gamma_{kji} e_k, \qquad \textit{postparastrophic},$$

$$M_5 \cdot \rho\sigma = \sum x_i y_j \gamma_{jki} e_k, \qquad \textit{antipostparastrophic},$$

$$M_6 \cdot \rho\sigma = \sum x_i y_j \gamma_{kij} e_k, \qquad \textit{antipreparastrophic}.$$

The antistrophic is also called the reciprocal. With the exception of the latter, the related algebras are in general not associative, even when the given algebra is associative. For an interpretation of $M_2, \cdots, M_6$ in terms of operations in the given algebra, see § 7.

2. If we let σ be in turn each of the units e, we can write out for each of the six algebras a general table of multiplication, consisting of r equations, thus:

$$M_1 \cdot \rho e_j = \sum e_k X^1_{jk}, \qquad X^1_{jk} = \sum x_i \gamma_{ijk},$$

$$M_2 \cdot \rho e_j = \sum e_k X^2_{jk}, \qquad X^2_{jk} = \sum x_i \gamma_{jik},$$

$$M_3 \cdot \rho e_j = \sum e_k X^3_{jk}, \qquad X^3_{jk} = \sum x_i \gamma_{ikj},$$

$$M_4 \cdot \rho e_j = \sum e_k X^4_{jk}, \qquad X^4_{jk} = \sum x_i \gamma_{kji},$$

$$M_5 \cdot \rho e_j = \sum e_k X^5_{jk}, \qquad X^5_{jk} = \sum x_i \gamma_{jki},$$

$$M_6 \cdot \rho e_j = \sum e_k X^6_{jk}, \qquad X^6_{jk} = \sum x_i \gamma_{kij}.$$

3. The number ρ in each case may be looked upon as an operator upon the following number, in fact it may be considered to be a linear substitution on the units. We may arrange the coefficients X in r rows and columns, and form a determinant for each case:

$$|X_{jk}| \qquad (j, k = 1, \cdots, r).$$

The column is indicated by the second subscript, the row by the first. This

determinant we shall indicate by Δ^1, Δ^2, Δ^3, Δ^4, Δ^5, Δ^6, in the respective cases, and if we need to indicate the number from which it arises, we shall write $\Delta(\rho)$ with the proper superscript.

4. If we write the multiplication table of each of the algebras as a square array, the multiplicands being the units e, arranged in order along the top of the square, and the multipliers the units e arranged in order vertically downwards, along the left side of the square, the product of e_j into e_k being in the square on the line j and in column k, then we have the entry

$$M_1 \cdot e_j e_k = \sum e_i \gamma_{jki}, \qquad M_2 \cdot e_j e_k = \sum e_i \gamma_{kji}, \qquad M_3 \cdot e_j e_k = \sum e_i \gamma_{jik},$$
$$M_4 \cdot e_j e_k = \sum e_i \gamma_{ikj}, \qquad M_5 \cdot e_j e_k = \sum e_i \gamma_{kij}, \qquad M_6 \cdot e_j e_k = \sum e_i \gamma_{ijk}.$$

If we compare these with the entries in the array of X's, we find that we can deduce one set of arrays from the other and reciprocally. In all the cases we change x_i to e_i and vice versa. We thus arrive at the simple rules following:

To derive the determinant of each algebra for the general number we change e to x in the multiplication table indicated, deriving Δ^1 from M_6, Δ^2 from M_3, Δ^3 from M_4, Δ^4 from M_2, Δ^5 from M_1, Δ^6 from M_5.

To derive the multiplication table of each algebra we change x to e in the determinant array indicated, deriving M_1 from Δ^5, M_2 from Δ^4, M_3 from Δ^2, M_4 from Δ^3, M_5 from Δ^6, M_6 from Δ^1.

5. The relationship of the X's to each other is seen more clearly if we use the notation of general vector algebra,* which gives us each X as an inner product:

$$X^1_{jk} = A_0 e_k(\rho e_j), \qquad X^2_{jk} = A_0 e_k(e_j \rho), \qquad X^3_{jk} = A_0 e_j(\rho e_k),$$
$$X^4_{jk} = A_0 \rho(e_k e_j), \qquad X^5_{jk} = A_0 \rho(e_j e_k), \qquad X^6_{jk} = A_0 e_j(e_k \rho).$$

Considering ρ as an operator (linear substitution) we may write ρ' for the transverse operator and then these expressions become (k indicating columns, j rows) the following inner products:

$$X^1_{jk} = A_0 e_k(\rho e_j), \qquad X^2_{jk} = A_0 e_k(e_j \rho), \qquad X^3_{jk} = A_0 e_k(\rho' e_j),$$
$$X^4_{jk} = A_0 e_k(\rho e'_j), \qquad X^5_{jk} = A_0 e_k(e'_j \rho), \qquad X^6_{jk} = A_0 e_k(e_j \rho').$$

6. From this notation we see that we may consider ρ as a linear vector operator in each case. The respective determinants become†

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* $A_0 \cdot \rho\sigma \equiv \Sigma x_i y_i$. This notation the author has used in other papers.

† $A_0 \cdot A_s \alpha_1 \cdots \alpha_s A_s \beta_1 \cdots \beta_s = (-1)^{\frac{1}{2}s(s-1)} \mid A_0 \cdot \alpha_i \beta_j \mid, \qquad (i, j = 1, \cdots, s),$
$A_0 \cdot A_s \alpha_1 \cdots \alpha_s A_s \phi\beta_1 \cdots \phi\beta_s = A_0 A_s \phi' \alpha_1 \cdots \phi' \alpha_s A_s \beta_1 \cdots \beta_s.$

$$\Delta^1 = tA_0\, A_r\, e_1 e_2 \cdots e_r\, A_r(\rho e_1)(\rho e_2)\cdots(\rho e_r),$$

$$\Delta^2 = tA_0\, A_r\, e_1 e_2 \cdots e_r\, A_r(e_1\rho)(e_2\rho)\cdots(e_r\rho),$$

$$\Delta^3 = tA_0\, A_r\, e_1 e_2 \cdots e_r\, A_r(\rho' e_1)(\rho' e_2)\cdots(\rho' e_r),$$

$$\Delta^4 = tA_0\, A_r\, e_1 e_2 \cdots e_r\, A_r(\rho e_1')(\rho e_2')\cdots(\rho e_r'),$$

$$\Delta^5 = tA_0\, A_r\, e_1 e_2 \cdots e_r\, A_r(e_1'\rho)(e_2'\rho)\cdots(e_r'\rho),$$

$$\Delta^6 = tA_0\, A_r\, e_1 e_2 \cdots e_r\, A_r(e_1\rho')(e_2\rho')\cdots(e_r\rho'),$$

where $(t = (-1)^{r(r-1)/2})$.

We see from these forms at once that we have identically $\Delta^3 = \Delta^1, \Delta^6 = \Delta^2$, and, by writing out the determinants, $\Delta^4 = \Delta^5$. Owing to the alternating character of the expressions A_r, they are not altered if we substitute $e_i + he_j$ for e_i, where h is any arbitrary scalar multiplier. But a repetition of similar substitutions would enable us to introduce a new basis, which we can represent by $\epsilon_1, \epsilon_2, \cdots, \epsilon_r$, where $\epsilon_i = \phi e_i$, ϕ being a linear substitution on the basis such that $|\phi| \neq 0$. That is to say, in place of each e in the forms above we may write ϵ; but it is to be noticed that the combinations with ρ are still formed under the original multiplication table. If now we desire to consider ρ as expressed in terms of the basis ϵ, and also make our inner product A_0 refer to the new defining units, we must remember that if A_0' refers to the basis ϵ just as A_0 to the basis e,

$$A_0'\, \xi\zeta = A_0\, \xi\phi'^{-1}\, \phi^{-1}\, \zeta,$$

where ϕ' is the transverse of ϕ. We will let $\phi\phi' = \psi$, which is self-transverse. Then

$$A_0'\, \xi\zeta = A_0\, \xi\psi^{-1}\, \zeta.$$

Also we notice that $A_0'\, \xi\theta(\zeta) = A_0'\, \zeta[\psi\theta'\, \psi^{-1}(\xi)]$ and $()\rho'$ for basis ϵ is $\psi[\{\psi^{-1}()\}\rho']$ for basis e; $\rho'()$ for ϵ is $\psi\{\rho'(\psi^{-1}[])\}$ for e.

The expression X_{jk} becomes under this change of basis (ρ now in terms of ϵ)

$$'X^1_{jk} = A_0'\, \epsilon_k(\rho\epsilon_j) = A_0\, \phi'^{-1}\, e_k(\rho\phi e_j)$$

and thus the determinant in terms of A_0', which we may represent by $'\Delta^1$, is

$$'\Delta^1 = A_0\, A_r\, \phi' e_1\, \phi' e_2 \cdots \phi' e_r\, A_r(\rho\phi'^{-1}\epsilon_1)(\rho\phi'^{-1}\epsilon_2)\cdots(\rho\phi'^{-1}\epsilon_r)(-)^{\frac{1}{2}r(r-1)},$$

which reduces to

$$|\phi| \cdot |\phi^{-1}| \cdot \Delta^1 = \Delta^1,$$

and it is evident that Δ^1 is invariant under a change of basis.

It is obvious that corresponding changes in the other determinants will leave them also invariant.

7. Returning to § 2 and using the last equations in § 5, and since

$\xi = \sum e_k A_0 e_k \xi$, we see that we may define the six multiplications in terms of ρ and the operator ρ' thus:

$$M_1 \cdot \rho e_j = \rho e_j, \qquad M_2 \cdot \rho e_j = e_j \rho, \qquad M_3 \cdot \rho e_j = \rho' e_j,$$
$$M_4 \cdot \rho e_j = \rho e_j', \qquad M_5 \cdot \rho e_j = e_j' \rho, \qquad M_6 \cdot \rho e_j = e_j \rho'.$$

When neither letter is accented we have the original strophic multiplication. The general formula of multiplication is then easily found by substituting the general number σ for e_j. There is obviously no essential difference between an algebra and the anti-algebra save as to which of the two numbers we consider to be the multiplicand. We may further omit the multiplication symbol M, since the three essentially different types of multiplication are now indicated definitely by writing

$$\rho\sigma, \qquad \rho'\sigma, \qquad \rho\sigma',$$

which are respectively the *strophic*, *preparastrophic*, *postparastrophic*, for ρ into σ, and are the corresponding anti-algebra for ρ by σ.

8. When the strophic is associative, we have

$$\rho \cdot \sigma \cdot () = (\rho\sigma) \cdot (), \qquad \rho' \cdot \sigma' \cdot () = (\sigma \cdot \rho \cdot)'() = (\sigma\rho)'(),$$

with similar equations for the post multiplication. Hence

$$(\rho\eta)'\xi = \eta'\rho'\xi \quad \text{or} \quad [()\eta]'\xi = \eta'[()'\xi].$$

Therefore we have the following equation in the determinants:

$$\Delta^5(\xi) \cdot \Delta^2(\eta) = \Delta^3(\eta) \cdot \Delta^5(\xi).$$

We might equally well have written $\xi[\eta()]' = \xi()' \cdot \eta'$ whence

$$\Delta^4(\xi) \cdot \Delta^1(\eta) = \Delta^6(\eta) \cdot \Delta^4(\xi).$$

It follows at once when $\Delta^4(\xi) = \Delta^5(\xi) \neq 0$ that we have identically

$$\Delta^1(\eta) = \Delta^3(\eta) = \Delta^2(\eta) = \Delta^6(\eta).$$

Further we have easily

$$\rho'(\eta'\xi) = (\eta\rho)'\xi \quad \text{or} \quad ()'(\eta'\xi) = [\eta()]'\xi,$$

and at once

$$\Delta^5(\eta'\xi) = \Delta^5(\xi) \cdot \Delta^1(\eta) = \Delta^4(\eta'\xi) = \Delta^2(\eta) \cdot \Delta^4(\xi).$$

Also

$$\xi\eta'\rho' = \xi(\rho\eta)' \quad \text{or} \quad (\xi\eta')()' = \xi[()\eta]',$$

giving

$$\Delta^4(\xi\eta') = \Delta'(\xi)\Delta^2(\eta) = \Delta^4(\eta'\xi).$$

These equations express Frobenius' fundamental theorems.

9. The general equation of the number ρ, whether for pre- or post-multi-

plication, has certain irreducible factors of determinant form corresponding to the invariant regions of ρ. Let one of these be Φ corresponding to an invariant region of order s, then we can also write

$$\Phi = \rho^s - m_1 \rho^{s-1} + m_2 \rho^{s-2} - \cdots (-)^s m_s e_1$$

and for a certain set of s units $e_2, \cdots, e_{s+1}$ we have for premultiplication,

$$\Phi e_i = 0, \qquad i = 2, \cdots, s+1,$$

and for a certain set of s units $a_2, \cdots, a_{s+1}$ we have $a_i \Phi = 0$.

The coefficient m_s may then be written in the form

$$\begin{aligned} m_s &= A_0 A_s (\rho e_2)(\rho e_3) \cdots (\rho e_{s+1}) A_s e_2 e_3 \cdots e_{s+1} (-)^{\frac{1}{2}s(s-1)} \\ &= A_0 A_s (a_2 \rho)(a_3 \rho) \cdots (a_{s+1} \rho) A_s a_2 a_3 \cdots a_{s+1} (-)^{\frac{1}{2}s(s-1)}. \end{aligned}$$

We will now operate upon both these with the generalized Hamiltonian operator* ∇, giving

$$\begin{aligned} \nabla m_s &= \textstyle\sum_{i=2}^{s+1} A_1 A_{s-1} (\rho e_2) \cdots (i) \cdots (\rho e_{s+1}) A_s e_2 e_3 \cdots e_{s+1} \cdot (-)^{\frac{1}{2}s(s-1)+i} \cdot e_i' \\ &= \textstyle\sum a_i' A_1 A_{s-1} (a_2 \rho) \cdots (i) \cdots (a_{s+1} \rho) A_s a_2 a_3 \cdots a_{s+1} \cdot (-)^{\frac{1}{2}s(s-1)+i} \\ &= \textstyle\sum_i \sum_t e_t e_i' A_0 A_{s-1} (\rho e_1) \cdots (i) \cdots (\rho e_{s+1}) A_{s-1} e_2 \cdots (t) \\ &\qquad \cdots e_{s+1} (-)^{\frac{1}{2}s(s-1)} \\ &= \textstyle\sum_i \sum_t e_i' e_t A_0 A_{s-1} (a_2 \rho) \cdots (i) \cdots (a_{s+1} \rho) A_{s-1} a_2 \cdots (t) \\ &\qquad \cdots a_{s+1} (-)^{\frac{1}{2}s(s-1)}. \end{aligned}$$

The (i) indicates an omitted factor.

Multiplying postparastrophically and preparastrophically into Φ we have, since $\Phi e_i = 0 = a_i \Phi$,

$$\nabla m_s \Phi' = 0 = \Phi' \nabla m_s.$$

Again, since for any number σ the first scalar invariant in the field that Φ annuls is given by

$$\begin{aligned} (-)^{\frac{1}{2}s(s-1)} m_1^{(s)} &= A_0 A_s (\sigma \lambda_1) \cdots \lambda_s A_s \lambda_1 \cdots \lambda_s \\ &\quad + A_0 A_s \lambda_1 (\sigma \lambda_2) \cdots \lambda_s A_s \lambda_1 \cdots \lambda_s + \cdots \end{aligned}$$

where the numbers λ are linearly independent, we have
$A_0 (\sigma \rho) \nabla m_s = m_1^{(s)} (\sigma) m_s^{(s)} = A_0 (\rho \sigma) \nabla m_s^{(s)} = A_0 \sigma (\Delta m_s^{(s)} \rho') = A_0 \sigma (\rho' \nabla m_s^{(s)})$.
Now $m_1 (\sigma)$ is linear in the coefficients of σ and we may therefore set

$$m_1^{(s)} (\sigma) = A_0 \sigma \kappa_s, \qquad \kappa_s = \textstyle\sum e_i m_1^{(s)} (e_i),$$

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* $\nabla = \Sigma e_i \dfrac{\partial}{\partial x_i}$.

and the numbers κ so defined for the different values of s are constant for the basis e. They usually vary for the change of basis. Making this substitution and remembering that σ is any number we have

$$m_s^{(s)}\,\kappa_s = \nabla m_s^{(s)}\,\rho' = \rho'\,\nabla m_s^{(s)}, \quad \text{or} \quad \kappa_s\,(m_s^{(s)}\,\rho'^{-1}) = \nabla m_s^{(s)}.$$

It is to be remembered that as the numbers λ were linearly independent this equation holds for those values of ρ that make $\rho e_2, \rho e_3, \cdots, \rho e_{s+1}$ linearly independent. In such case $m_s^{(s)} \neq 0$, and we have

$$\kappa_s = \nabla \log m_s^{(s)}\,\rho' = \rho'\,\nabla \log m_s^{(s)}.$$

By parastrophic multiplication we see that we have also

$$m_s^{(s)}\,\kappa_s\,\xi' = \nabla m_s^{(s)}\,(\xi\rho)', \qquad m_s^{(s)}\,\xi'\,\kappa_s = (\rho\xi)'\,\nabla m_s^{(s)}.$$

10. If we multiply the different κ's for the p different irreducible factors of the general equation by arbitrary constants c and add we have

$$\sum c_s\,\kappa_s = \rho'\,\nabla \log\,(m_{s_1}^{(1)f_1} \cdot m_{s_2}^{(2)f_2} \cdots m_{s_p}^{(p)f_p}).$$

This cannot vanish identically for arbitrary values of the constants without the product in the parenthesis being constant in the coefficients of ρ, which is absurd. Hence the different numbers κ are linearly independent.

What has been said above for certain irreducible factors of the general equation will hold equally well for products of these factors, no factor being repeated and for the regions corresponding. If however we study an irreducible factor raised to a power not higher than its power in the general prelatent equation, or the general postlatent equation, we see from the formula above that we have no new κ but simply a multiple of the original κ, which multiple is κ times the power of the irreducible factor. A similar statement will hold for the numbers deduced from combinations of the irreducible factors, and we find thus combinations of the form $\sum \mu_s\,\kappa_s$, where the numerical values μ are the indices of the powers of the factors. When these are the indices in the general prelatent equation we arrive at a special number we will call κ_1. It is evident that

$$\kappa_1 = \sum e_i\,m_1'\,(e_i) = \sum A_0\,e_j\,(e_i\,e_j)\cdot e_i = \sum e_j\,e_j'.$$

There is likewise a number arising from the postlatent equation which will be called κ_2,

$$\kappa_2 = \sum e_i\,m_1''\,(e_i) = \sum e_i\,A_0\,e_j\,(e_j\,e_i) = \sum e_j'\,e_j.$$

From these we have easily

$$m_1'\,(\rho) = A_0\,\rho\kappa_1, \qquad m_1''\,(\rho) = A_0\,\rho\kappa_2.$$

Further $\kappa_1 = \nabla\rho'$, $\kappa_2 = \rho'\nabla$. We have in a manner entirely similar to the process above

$$\kappa_1 = \nabla \log m'_r\, \rho', \qquad \kappa_2 = \rho' \nabla \log m''_r.$$

These forms for κ_1, κ_2 are particularly simple. Setting the expressions for κ_1 equal and those for κ_2 equal we have easily

$$\nabla(\rho'/m'_r) = 0 = (\rho'/m''_r)\nabla.$$

We cannot assume from these that the expressions in the parentheses are constants for the vanishing is due partly to the hypercomplex multiplication and not wholly to the differentiation.

From the definition we see that if we change the basis we have

$$A_0\,\sigma\kappa = A'_0\,\sigma\psi\kappa = A'_0\,\sigma\kappa_\psi,$$

where κ_ψ signifies that we find the number κ for the new basis. Comparing these equations we have at once $\kappa_\psi = \psi\kappa$, since σ is any number. This equation holds for all the numbers κ.

11. It is clear from the last equations in § 9 that the only number that will give $\kappa\xi' = \kappa$ or $\xi'\kappa = \kappa$, for any given irreducible factor, will be the number for which for that latent region $\rho\xi = \rho = \xi\rho$ for all values of ρ in the region which have a non-vanishing m, that is it will be the partial modulus of the latent region. Again, since $\nabla m_s \Phi' = 0$, it follows that also

$$\kappa_s \Phi' = 0 = \Phi' \kappa_s.$$

The number κ is thus entirely within the sth latent region of ρ', either pre or post.

12. It is evidently an interesting question to answer as to when ρ' is itself not simply an operator determined by ρ, but a number of the system. If we let the semi-simple subalgebra of A be B and the maximal invariant nilpotent system be N, then $A = B + N$. Now let us suppose that ρ is a number in N and has a transverse $\rho' = \tau$ in the algebra A. Then since the product of ρ into any number ξ of the system is in N, we have if β is in B

$$A_0\,\beta(\rho\xi) = 0 = A_0\,\xi(\rho'\beta) = A_0\,\xi(\tau\beta) \qquad \text{(for any } \xi).$$

Hence $\tau\beta = 0$, for β in B. But the modulus e_1 is in B and thus $\tau e_1 = \tau = 0$, and therefore $A_0\,\beta(\tau\xi) = 0 = A_0\,\xi(\rho\beta)$, and $\rho = 0$.

13. It follows at once that if every number of the algebra has a transverse in the algebra, then the algebra has no maximal invariant nilpotent system and is a Dedekind algebra.

This furnishes an extremely simple proof that every finite group defines a Dedekind algebra of a special kind, a Frobenius algebra. For if the operators

of the group are represented by $e_1, \cdots e_n$, then we may represent the product $e_i e_j$ by e_{ij}; and if $\rho = \sum x_i e_i$ we have

$$A_0 \sum z_i e_i (\sum x_j e_j \sum t_k e_k) = A_0 \sum z_i e_i (\sum x_j t_k e_{jk}) = \sum z_{jk} x_j t_k$$

$$A_0 \sum t_k e_k (\sum x_j e_j^{-1} \sum z_i e_i) = A_0 \sum t_k e_k \sum x_j z_i e_{ji}^{-1} = \sum z_i x_j t_{ji}^{-1} = \sum z_{jk} x_j t_k .$$

It is obvious then that the transverse of $\rho = \sum x_i e_i$ is in the algebra and is

$$\tau = \rho' = \sum x_i e_i^{-1} .$$

Likewise in the semi-simple system let the units be λ_{pq} where $\lambda_{pq} \lambda_{rs} = \delta_{qr} \lambda_{ps}$, the symbol δ_{qr} having the value 0 if $q \neq r$, and 1 if $q = r$. Then we have

$$A_0 \sum z_{pq} \lambda_{pq} (\sum x_{uv} \lambda_{uv} \sum t_{gh} \lambda_{gh}) = A_0 \sum z_{pq} \lambda_{pq} (\sum x_{uv} t_{vh} \lambda_{uh}) = \sum z_{uh} x_{uv} t_{vh} ,$$

$$A_0 \sum t_{gh} \lambda_{gh} (\sum x_{uv} \lambda_{uv} \sum z_{pq} \lambda_{pq}) = A_0 \sum t_{gh} \lambda_{gh} (\sum x_{pv} z_{pq} \lambda_{vq})$$

$$= \sum t_{vq} x_{pv} z_{pq} = \sum z_{uh} x_{uv} t_{vh} .$$

Thus the transverse of ρ exists and is

$$\rho' = \tau = \sum x_{vu} \lambda_{uv} .$$

It would be interesting to apply to these two examples the preceding theorems, but we will content ourselves only with pointing out that for the group

$$\kappa_1 = \nabla \rho' = \sum e_i e_i^{-1} = n e_1 = \kappa_2 ,$$

that is, n times the identical operator. For the semi-simple or Dedekind algebra

$$\kappa_1 = \kappa_2 = \nabla \rho' = \sum \lambda_{vu} \lambda_{uv} = \sum w_j \eta_j ,$$

where w_j^2 and η_j are respectively the orders of the integral subalgebras and their partial moduli.

14. If the basis of the algebra is subjected to the non-singular transformation ϕ, and $\phi \phi' = \psi$, we have seen already that any operator θ' becomes $\psi \theta' \psi^{-1}$, hence as operator, ρ' on the new basis would be $\psi \rho' \psi^{-1}$ in terms of the old basis. Hence the transverse of ρ on the new basis in terms of the units of the old basis is not τ but becomes

$$\tau_\epsilon = \psi [\tau \psi^{-1} e_1]$$

where e_1 is the modulus, and $\tau_\epsilon \sigma = \psi [\tau \psi^{-1} \sigma]$. Hence

$$\tau_\epsilon \psi e_1 = \psi \tau , \qquad \tau_\epsilon \psi e_i = \psi (\tau e_i) .$$

These equations are not satisfied for every value of ϕ, nor ψ, but only for those values which permit of the simultaneous existence of the equations. In other words, if we let τ run through the entire range of numbers in the algebra we must have in all cases as a condition on ψ

$$\psi \tau = \tau_\epsilon \psi e_1 = \psi (\tau \psi^{-1} e_1) \psi e_1 = \psi e_1 (\psi^{-1} e_1 \cdot \tau) .$$

We may state thus the condition on ψ from this equation: If we form the multiplication table for the original basis, and then operate on multipliers, multiplicands, and products all alike by either $(\psi e_1)^{-1}\psi(\,)$ or $\psi(\,)(\psi e_1)^{-1}$ the new table must be still a correct multiplication table for the algebra.

It is to be noticed that ψe_1 is a multiple of $\kappa_1 = \kappa_2$ for the new basis. It is also evident that $\kappa = \kappa_1 = \kappa_2$ must have an inverse. Any operator ϕ which leads to ψ such that $\psi e_1 = \kappa$ does not have an inverse, is not permissible. The same algebra may therefore be such that it permits a transverse for every number on some bases but not on others.

15. When $\rho' = \tau$ is in the algebra the preparastrophic and the postparastrophic multiplications can be defined in terms of the strophic multiplication.

For example, in the case of an abstract group, parastrophic multiplication by ρ is strophic multiplication by $\sum x_i e_i^{-1}$.

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